"Comprehensive Support for Students in Mathematics subject seeking to Overcome Past Setbacks."

MATHEMATICS AND STATISTICS

Std. - XII

(Commerce)

Part - II

State Council of Educational Research and Training, Maharashtra, Pune

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'Comprehensive Support for Students in Mathematics subject seeking to Overcome Past Setbacks.'

Specialized Mathematics Study Materials for HSC Students

Subject : Mathematics and statistics (Commerce) Code : 88

OBJECTIVES OF THE BOOKLET

This booklet is prepared for the help of the students who will be appearing for the Supplementary Examination to be held in July 2024 and thereafter too. It is prepared as such students could not score the minimum score to pass in the written examination held in February 2024.

This booklet is designed to boost the confidence of the students. It will definitely help them to score good marks in the forthcoming examination. It will be a great support for the students who lack behind others.

It is prepared in a systematic and easiest way by the expert teachers. The students are aware of the textbook as well as the examination pattern (MCQ's, Fill in the Blanks, True/False, 3 Marks and 4 Marks, Activities questions). Still, this booklet elaborates every segment in detail. It considers the level of the students.

By studying as suggested in the booklet, we are quite sure that the students will be able to practice a lot with given guidelines. They will score and step into the world of success.

The main objectives can be summarized as under :

1) To facilitate the essential study material to the students to confidently face the HSC Board Examination.
2) To help every average and the below average student to achieve 100% success at the HSC Board Examination.
3) To motivate the below average students to score more than their expectation in the Mathematics Subject which they find as most difficult.
4) To include tools and exercises that allow students to evaluate their own progress and understand their improvement areas.
5) To help the teachers to reach out to students who struggle to pass in the Mathematics subject at the HSC Board Exam with the help of this material.
6) Each chapter in the booklet contains important concepts in short.
7) Based on these concepts simple solved examples are given.
8) Practice questions with hints and answers are given.
9) Two practice question papers will definitely help students.

INTRODUCTION

Dear Students,

It does not matter if you did not score well in the regular examination held in February 2024. Remember,"every setback is a setup for a comeback." Your previous attempt must have taught you something valuable. We believe in your potential to overcome this hurdle and excel in your upcoming exams.

After a comprehensive analysis of the results, SCERT, Pune has taken an initiative for the upliftment of students who could not achieve the minimum passing score. It was found that some fundamental concepts were not clear to the students. Hence, a significant effort was made to prepare this booklet.

This booklet is designed specifically for those who did not achieve the desired results in their previous Mathematics exam. We understand that facing a setback can be challenging, but it also presents an invaluable opportunity for growth and learning. Our goal with this booklet is to provide you with comprehensive resources and targeted exercises to help you strengthen your understanding of key mathematical concepts. We have carefully curated the content to address common areas of difficulty and to reinforce fundamental principles essential for success in Mathematics.

This booklet will help you to prepare for the supplementary examination to be held in July 2024. Through a combination of clear explanations, step-by-step problem-solving strategies, and ample practice questions, we aim to build your confidence and competence in the subject. Remember, perseverance and a positive mindset are crucial as you work through this material.

Use this booklet diligently, seek help when needed, and stay committed to your studies. With dedication and effort, you can turn this experience into a stepping stone toward academic success. This resource will also prove to be extremely useful for teachers as they assist students in preparing for the supplementary examination. It will boost your confidence to appear for the exam once again. New students in the coming years can also benefit from this booklet.

Best wishes for your journey ahead.

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Part - II

1. Commission, Brokerage and Discount

Important Terms :

Commission and Brokerage Agent :

When transactions like sale, purchase, auction etc. are done through some middlemen, such middlemen are called agents.

Principal :

Principal refers to an individual party or parties participating in a transaction.

Commission :

The charges paid to an agent for doing the work on behalf of principal is called commission.

Different types of agents :

- **Commission Agents :** A commission agent is a person who buys or sells goods on behalf of his principal and gets commission for his service.
- **Broker :** A broker is an agent who brings together the buyer and the seller for the purpose of purchase or sale. This commission is called brokerage and is charged to both the parties.
- **Auctioneer :** An auctioneer is an agent who sells goods by auction. He sells goods to the highest bidder.
- **Factor :** A factor is an agent who is given the possession of goods and enters a contract for sale in his/her own name.
- **Del Credere Agent :** A del credere agent gives guarantee to his principal that the party to whom he/she sells the goods will pay the sale price of goods.

Trade Discount and Cash Discount :

Discount is the reduction in the price of an article, allowed by the seller to the purchaser

There are two types of Discount :

1) **Trade discount :** Trade discount is allowed by one trader to another. It is given on the catalogue price, list price or market price of the goods.
2) **Cash discount :** Cash discount is allowed in consideration of ready cash payment.

Invoice Price	: List price (Catalogue price)	– Trade discount.
Selling Price / Net Selling Price	: Invoice price	– Cash discount
Profit	: Net selling price	– Cost price
Loss	: Cost price	– Net selling price

Discount :

- **Present worth, sum due, true discount :**

  Present worth + True discount = Sum due

  i.e. P.W. + T.D. = S.D.

  $$\text{T.D.} = \frac{\text{P.W.} \times n \times r}{100}$$

  where P.W. is the principal or the present worth, n is period of the bill in years, r is the rate of interest per annum.

- **Drawer and Drawee :** A person who draws the bill is called the drawer. A person on whom the bill is drawn is called as Drawee.

- **Date of bill and Face value :** The date on which the bill is drawn is called as 'date of bill'. The amount for which the bill is drawn is called face value (F.V.) of the bill. It is the sum due on the present worth.

- **Nominal Due Date and Legal Due Date :** The date on which the period of bill expires is called the **Nominal due date**. The buyer has to make the payment to the seller on this date.

  The date obtained after adding 3 days of grace to the nominal due date is known as the **Legal due date**.

- **Discounting a Bill :** The facility available at the bank or with an agent who can discount a bill and pay the amount to the drawer before the legal due date is called Discounting a Bill.

- **Banker's Discount, Cash Value, Banker's Gain :** When a bill is discounted in a bank, the banker will deduct the amount from the face value of the bill at the given rate of interest for the period from the date of discounting to the legal due date and pay the balance to the drawer. This amount is known as Banker's Discount (B.D).

  The amount paid to the holder of the bill after deducting banker's discount is known as Cash Value (C.V) of the bill paid on the date of discounting. The difference between the banker's discount and the true discount is called Banker's Gain (B.G). It is equal to the interest on true discount.

Abbreviations :

- Present worth : P.W. or P
- True Discount : T.D.
- Banker's Discount : B.D.
- Sum Due/Face Value : S.D. or F.V.
- Banker's Gain : B.G.
- Cash Value : C.V.

Notation :

- Period (in Years) : n
- Rate of Interest (p.a.) : r

List of Formula :

1) S.D. : P.W. + T.D.

2) T.D. : $\dfrac{\text{P.W.} \times n \times r}{100}$

3) B.D. : $\dfrac{\text{S.D.} \times n \times r}{100}$

4) B.G. : B.D. − T.D.

5) B.G. : $\dfrac{\text{T.D.} \times n \times r}{100}$

6) Cash Value : S.D. − B.D.

SOLVED EXAMPLES :

Ex. 1) A merchant gives his agent 5% ordinary commission plus 2% del credere commission on sale of goods, worth Rs. 55,000/-. How much does the agent receive? How much does the merchant receive?

Solution : Agent's Commission at 5% $= 55{,}000 \times \dfrac{5}{100}$

= Rs. 2750

Rate of delcredere = 2%

Amount of delcredere $= 55{,}000 \times \dfrac{2}{100}$

= Rs. 1100

Agent's Total Commission = 2750 + 1100

= Rs. 3850

Merchant receives = 55000 − 3850

= Rs. 51150

Ex. 2) The price of a refrigerator is Rs. 47,000. An agent charges commission at 6% and earns Rs. 42,300. Find the number of refrigerators.

Solution : The Price of a refrigerator = Rs. 47,000/–

Rate of commission = 6%

commission for one refrigerator $= 47{,}000 \times \dfrac{6}{100}$

= Rs. 2820

Total Commission earned = Rs. 42,300 (given)

$\therefore$ Number of total refrigerators sold

$$= \frac{\text{Total Commission}}{\text{commission per refrigerator}}$$

$$= \frac{42300}{2820}$$

$$= 15$$

$\therefore$ 15 refrigerators were sold

Ex. 3) A sales representative gets fixed monthly salary plus commission based on the sales. In two successive months he received Rs. 23,500 and Rs. 24,250 on the sale of Rs. 70,000 and Rs. 85,000 respectively. Find his monthly salary and the rate of commission on sales.

Solution : Income of sales representative

= Salary + commission on the sales

23,500 = Salary + commission on Rs. 70,000 ......(1)

24,250 = Salary + commission on Rs. 85,000 ......(2)

subtracting (1) from (2) we get

750 = commission on Rs. 15,000

$$\text{Rate of Commission} = \frac{100 \times 750}{15,000}$$

$$= 5$$

Rate of Commission = 5%

$$\therefore \text{Commission on Rs. } 70,000 = 70,000 \times \frac{5}{100}$$

$$= 3,500$$

Substituting in equation (1), we get

23,500 = Salary + 3,500

Salary = 23,500 − 3,500

= Rs. 20,000

Ex. 4) A salesman receives 8% commission on the total sales. If his sales exceeds Rs. 20,000 he receives an additional commission at 2% on the sales over Rs. 20,000/-. If he receives Rs. 7,600 as commission, find his total sales.

Solution : Let the total sales be Rs. x.

Commission at 8% on total sales

$$= x \times \frac{8}{100} = \frac{8x}{100}$$

Sales exceeding Rs. 20,000 = $x - 20{,}000$

Commission at 2% on excess sales

$$= (x - 20000) \times \frac{2}{100}$$

But total commission earned = Rs.7,600

$$\frac{8x}{100} + \frac{2\,(x - 20{,}000)}{100} = 7{,}600$$

$$8x + 2x - 40000 = 7600 \times 100$$

$$10x = 8{,}00{,}000$$

$$x = 80{,}000$$

His total sales is Rs. 80,000

Ex. 5) M/s. Saket Electronics is given 15% trade discount and 5% cash discount on purchase of television sets by the distributor. Find the total discount availed if M/s. Saket Electronics purchases TV sets worth Rs. 12,00,000 from the distributor.

Solution : Discount at 15% on Rs. 12,00,000

$$= 12{,}00{,}000 \times \frac{15}{100}$$

= Rs. 1,80,000

$\therefore$ Invoice price of T.V. Sets

= 12,00,000 − 1,80,000

= Rs. 10,20,000

Now cash discount is given on Rs. 10,20,000

Cash discount at 5% on Rs.10,20,000

$$= 10{,}20{,}000 \times \frac{5}{100}$$

= Rs. 51,000

$\therefore$ Total discount availed = 1,80,000 + 51,000

= Rs. 2,31,000

Ex. 6) A motor bike is marked at Rs. 50,000. A retailer allows a discount at 16% and still gains 20% on the cost. Find purchase price of the retailer.

Solution : List price of the motor bike = Rs 50,000

Discount at 16% on Rs. 50,000.

$= 50000 \times \frac{16}{100}$

$= \text{Rs. } 8{,}000$

$\therefore$ Selling price $= 50{,}000 - 8{,}000$

$= \text{Rs. } 42{,}000$

In case the purchase price is Rs. 100, the selling price is Rs. 120

$\therefore$ For selling price = 42,000

the purchase price $= \frac{100 \times 42{,}000}{120}$

$= \text{Rs. } 35{,}000$

$\therefore$ Purchase Price of the motor bike is Rs. 35,000.

Ex. 7) If the present worth of a bill due six months hence is Rs. 23,000 at 8% p.a., What is sum due?

Solution : P.W. = 23,000, $r = 8\%$

$n = 6 \text{ months} = \frac{1}{2} \text{ year}$

$\text{T.D.} = \frac{\text{P.W.} \times n \times r}{100}$

$= \frac{23000 \times \frac{1}{2} \times 8}{100}$

$= \text{Rs. } 920$

Now S.D. $= \text{P.W.} + \text{T.D.}$

$= 23{,}000 + 920$

$= \text{Rs. } 23{,}920$

$\therefore$ The sum due is Rs. 23,920

Ex. 8) The present worth of sum of Rs. 8,268 due 8 months hence is Rs. 7,800. Find the rate of interest.

Solution : S.D. = Rs. 8,268, P.W. = 7,800,

$n = \frac{8}{12}$ years

Now T.D. $= \text{S.D.} - \text{P.W.}$

$= 8{,}268 - 7{,}800$

$= 468$

$$\text{T.D.} = \frac{\text{P.W.} \times n \times r}{100}$$

$$468 = \frac{7800 \times \frac{8}{12} \times r}{100}$$

$$468 = 78 \times \frac{2}{3} \times r$$

$$r = \frac{468 \times 3}{78 \times 2}$$

$= 9$

$\therefore$ Rate of interest is 9%

Ex. 9) A bill of Rs. 10,100 drawn on 14th January for 5 months was discounted on 26th March. The customer was paid Rs. 9,939.25. Calculate the rate of interest.

Solution : F.V = Rs.10,100, C.V = Rs.9,939.25

Banker's Discount (B.D) = F.V − C.V

= 10,100 − 9,939.25

= 160.75

Date of drawing – 14^{th} January

Period – 5 months

Nominal due date – 14^{th} June

Legal due date – 17^{th} June

Date of discounting – 26^{th} March

Number of days from the date of discounting to legal due date.

March	April	May	June	Total
05	30	31	17	83

$\therefore$ B.D. = interest on F.V. for 83 days at r %

$$160.75 = 10,100 \times \frac{83}{365} \times \frac{r}{100}$$

$$\therefore \quad r = \frac{16075 \times 365}{10100 \times 83}$$

$= 6.99$

$\therefore$ Rate of interest is 6.99% ≈ 7%

Ex. 10) Find the true discount, banker's discount and banker's gain on a bill of Rs. 64,800 due 3 months hence discounted at 5% p.a.

Solution : S.D. = 64,800, n = 3 months = $\frac{3}{12} = \frac{1}{4}$ years, r = 5% p.a.

$$\text{Now B.D.} = \frac{\text{S.D.} \times n \times r}{100}$$

$$= 64{,}800 \times \frac{1}{4} \times \frac{5}{100}$$

$$= \text{Rs. } 810$$

Let T.D. = Rs. x

B.D. = T.D. + Interest on T.D. for $\frac{1}{4}$ year at 5% p.a.

$$810 = x + \left(x \times \frac{1}{4} \times \frac{5}{100}\right)$$

$$= x + \frac{x}{80}$$

$$= \frac{81x}{80}$$

$$\therefore x = \frac{810 \times 80}{81}$$

$$= \text{Rs. } 800$$

Banker's gain = banker's discount − true discount

i.e. B.G. = B.D. − T.D.

= 810 − 800

= Rs.10

∴ Banker's gain is Rs.10

Ex. 11) A banker's discount calculated for 1 year is 13.5 times its banker's gain. Find the rate of interest

Solution : Let the banker's gain = Rs. x

∴ B.D. = 13.5 × x

= 13.5 x

Now B.G. = B.D − T.D.

x = 13.5x − T.D.

T.D. = 12.5 x

But B.G. = Interest on T.D. for 1 year

$$= \frac{\text{T.D.} \times n \times r}{100}$$

$$x = \frac{12.5\, x \times 1 \times r}{100}$$

$$100\, x = 12.5\, xr$$

$$r = 8$$

$\therefore$ Rate of interest is 8% p.a.

MULTIPLE CHOICE QUESTIONS

I) Choose the correct alternative.

1) An agent who gives guarantee to his principal that the party will pay the sale price of goods is called ....

a) Auctioneer b) Del Credere Agent

c) Factor d) Broker

2) An agent who is given the possession of goods to be sold is known as ....

a) Factor b) Broker

c) Auctioneer d) Del Credere Agent

3) The date on which the period of the bill expires is called ....

a) Legal Due Date b) Grace Date

c) Nominal Due Date d) Date of Drawing

4) The payment date after adding 3 days of grace period is known as ....

a) The legal due date b) The nominal due date

c) Days of grace d) Date of drawing

5) The sum due is also called as ....

a) Face value b) Present value

c) Cash value d) True discount

6) P is the abbreviation of ....

a) Face value b) Present worth

c) Cash value d) True discount

7) Banker's gain is simple interest on ....

a) Banker's discount b) Face Value

c) Cash value d) True discount

8) The marked price is also called as ....

a) Cost price b) Selling price

c) List price d) Invoice price

9. When only one discount is given then ....

a) List price = Invoice price b) Invoice price = Net selling price

c) Invoice price = Cost price d) Cost price = Net selling price

10. The difference between face value and present worth is called ....

a) Banker's discount b) True discount

c) Banker's gain d) Cash value

II) Fill in the blanks.

1) A person who draws the bill is called ....
2) An .... is an agent who sells the goods by auction.
3) Trade discount is allowed on the .... price.
4) The banker's discount is also called ....
5) The banker's discount is always .... than the true discount.
6) The difference between the banker's discount and the true discount is called............ .
7) The date by which the buyer is legally allowed to pay the amount is known as........ .
8) A ....is an agent who brings together the buyer and the seller.
9) If buyer is allowed both trade and cash discounts, discount is first calculated on price.

III) State whether each of the following is True or False.

1) Broker is an agent who gives a guarantee to seller that the buyer will pay the sale price of goods.
2) Cash discount is allowed on list price.
3) Trade discount is allowed on catalogue price.
4) The buyer is legally allowed 6 days grace period.
5) The date on which the period of the bill expires is called the nominal due date.

6) The difference between the banker's discount and true discount is called sum due.

7) The banker's discount is always lower than the true discount.

8) The bankers discount is also called as commercial discount.

Exercise - 1.1

1) An agent charges 12% commission on the sales. What does he earn if the total sale amounts to Rs. 48,000? What does the seller get?

2) Ms. Saraswati was paid Rs. 88,000 as commission on the sale of computers at the rate of 12.5%. If the price of each computer was Rs. 32,000, how many computers did she sell?

3) Mr. Pavan is paid a fixed weekly salary plus commission based on percentage of sales made by him. If on the sale of Rs. 68,000 and Rs. 73,000 in two successive weeks, he received in all Rs. 9,880 and Rs. 10,180, Find his weekly salary and the rate of commission paid to him.

4) Three cars were sold through an agent for Rs. 2,40,000, Rs. 2,22,000 and Rs. 2,25,000 respectively. The rates of commission were 17.5% on the first, 12.5% on the second. If the agent overall received 14% commission on the total sales, find the rate of commission paid on the third car.

5) An agent charges 10% commission plus 2% delcreder. If he sells goods worth Rs. 37,200, find his total earnings.

Exercise - 1.2

1) What is the present worth of a sum of Rs. 10,920 due six months hence at 8% p.a. simple interest?

2) What is sum due of Rs. 8,000 due 4 months hence at 12.5% simple interest?

3) Find the true discount, Banker's discount and Banker's gain on a bill of Rs. 4,240 due 6 months hence at 9% p.a.

4) A bill of Rs. 8,000 drawn on 5th January 1998 for 8 months was discounted for Rs. 7,680 on a certain date. Find the date on which it was discounted at 10% p.a.

5) A bill was drawn on 14th April for Rs. 7,000 and was discounted on 6th July at 5% p.a. The Banker paid Rs. 6,930 for the bill. Find period of the bill.

6) A bill of Rs. 51,000 was drawn on 18th February 2010 for 9 months. It was encashed on 28th June 2010 at 5% p.a. Calculate the banker's gain and true discount.

Activities

1) The value of the goods sold = Rs. x

Commission @ 7.5% on first Rs. 10,000 = Rs. ☐

Commission @ 5% on the balance

Rs. $(x - 10{,}000) = \frac{5}{100} \times$ ☐

= Rs. ☐

An Agent remits Rs. 33,950 to his Principal

$\therefore\ x -$ ☐ $-$ ☐ $= 33{,}950$

$\frac{95x}{100} = 33{,}950 +$ ☐

$\frac{19x}{\square} = 34{,}200$

$x =$ Rs. ☐

2) Face Value (S) = Rs. 4,015 r = 8% p.a.

Date of drawing bill	= 19th January 2018
Period of the bill	= 8 months
Nominal Due date	= ☐
Legal Due date	= 22nd September 2018
Date of discounting the bill	= 28th February 2018

Number of days from date of discounting to legal due date

March	**April**	**May**	**June**	**July**	**Aug.**	**Sept.**	**Total**
31	☐	☐	30	☐	31	☐	206 days

$\therefore\ \text{B.D.} = \frac{\text{s. n. r.}}{100} \qquad = 4015 \times \frac{206}{365} \times \frac{8}{100}$

= ₹ 181.30

$\therefore$ C.V. = 4015 − ☐

= ₹ 3833.70

■■■

2. Insurance and Annuity

The verb "to insure" means to arrange for compensation in the event of damage or total loss of property or injury or the death of someone, in exchange of regular payments to a company or to the state. The word "insurance" means creation

of some security or monetary protection against a possible damage or loss. Insurance is a legal contract between an insurance company (insurer) and a person covered by the insurance (insured). An insurance policy is a legal document of the contract or agreement between the two parties, the insured and the insurer.

Insurance is of two types : Life Insurance and General Insurance.

1) **Life Insurance :** A person who wishes to be insured for life agrees to pay the insurance company a certain amount of money periodically. This amount is called the premium. The period of the payment can be a month, a quarter, half-year, or a year. In return, the insurance company agrees to pay a definite amount of money in the event of death of the insured or maturity of the policy, that is, at end of the contract period. This amount is called the policy value.

2) **General Insurance :** General insurance covers all types of insurance except life insurance. General insurance allows a person to insure properties like buildings, factories, and godowns containing goods against a possible loss (total or partial) due to fire, flood, earthquake, etc.

   Vehicles can be insured to cover the risk of possible damage due to accidents. In case of loss or damage, the insurance company pays compensation in the same proportion that exists between the policy value and the property value. All contracts of general insurance are governed by the principle of indemnity, which states that an insured may not be compensated by the insurance company in an amount exceeding the insured's economic loss. As a result, an insured person cannot make profit from an insurance policy.

3) **Fire Insurance :** It Fire insurance is property insurance that covers damage and losses caused by fire to property like buildings, godowns containing goods, factories, etc. It is possible to insure the entire property or only its part. The value of the property is called Property Value. The value of the insured part of property is called Policy Value. The amount paid to the insurance company to insure the property is called premium.

$$\text{Premium} = \text{Rate of Premium} \times \text{Policy Value}$$

The period of a fire insurance policy is one year and the premium is expressed as percentage of the value of the insured property.

In case of damage to the property due to fire, the insurance company agrees to pay compensation in the proportion that exist between policy value and property value. The value of the damage is called "loss" and the amount that the insured can demand under the policy is called claim.

$$\text{Claim} = \text{Loss} \times \frac{\text{Policy Value}}{\text{Property Value}}$$

Accident Insurance :

Personal accident insurance is a policy that can reimburse your medical costs, provide compensation in case of disability or death caused by accidents. Accident insurance allows insuring vehicles like cars, trucks, two wheelers, etc. against to a vehicle due to accidents. This policy also covers the liability of the insured person to third parties involved in the accident. The period of such policies is one year

Marine Insurance :

Marine Insurance covers goods, freight, cargo, etc. against loss or damage during transit by road, rail, sea or air. Shipments are protected from the time they leave the seller's warehouse till the time they reach the buyer's warehouse. Marine insurance offers complete protection during transit goods and compensates in the events of any loss.

The party responsible for insuring the goods is determined by the sales contract. The amount of claim is calculated by the same method that is used in the case of fire insurance.

Premium is paid on insured value.

Agent's commission is paid on premium..

$$\text{Claim} = \text{Loss} \times \frac{\text{Policy Value}}{\text{Property Value}} \times \text{loss}$$

Let's Study :

2.2 Annuity :

When you deposit some money in a bank, you are entitled to receive more money (in the form of interest) from the bank than you deposit, after a certain period of time. Similarly, when people borrow money for a certain period of time, they pay back more money (again, in the form of interest). These two examples show how money has a time value. A rupee today is worth more than a rupee after one year. The time value of money explains why interest is paid or earned. Interest, whether it is on a bank deposit or a loan, compensates the depositor or lender for the time value of money. When financial transactions occur at different points of time, they must be brought to a common point in time to make them comparable. Consider the following situation. Ashok deposits Rs.1000 every year in his bank account for 5 years at a compound interest rate of 10 per cent per annum. What amount will Ashok receive at the end of five years? In other words, we wish to know the future value of the money Ashok deposited annually for five years in his bank account.

Assuming that Rs.1000 are deposited at end of every year, the future value is given by

$$1000\left[\left(1+\frac{1}{10}\right)^4+\left(1+\frac{1}{10}\right)^3+\left(1+\frac{1}{10}\right)^2+\left(1+\frac{1}{10}\right)^1+1\right] = Rs.\ 6105.1$$

In the same situation, the present value of the amount that Ashok deposits in his bank account is given by

$$\frac{100}{(1.10)^1} + \frac{100}{(1.10)^2} + \frac{100}{(1.10)^3} + \frac{100}{(1.10)^4} + \frac{100}{(1.10)^5} = Rs.\ 3790.78$$

We also come across a situation where a financial company offers to pay Rs. 8,000 after 12 years for Rs.1000 deposited today.

In such situations, we wish to know the interest rate offered by the company.

Studies of this nature can be carried out by studying Annuity.

An annuity is a sequence of payments of equal amounts with a fixed frequency. The term "annuity" originally referred to annual payments (hence the name), but it is now also used for payments with any frequency. Annuities appear in many situations: for example, interest payments on an investment can be considered as an annuity. An important application is the schedule of payments to pay off a loan. The word "annuity" refers in everyday language usually to a life annuity. A life annuity pays out an income at regular intervals until you die. Thus, the number of payments that a life annuity makes is not known. An annuity with a fixed number of payments is called an annuity certain, while an annuity whose number of payments depend on some other event (such as a life annuity) is a contingent annuity. Valuing contingent annuities requires the use of probabilities.

Definition :

An annuity is a series of payments at fixed intervals, guaranteed for a fixed number of years or the lifetime of one or more individuals. Similar to a pension, the money is paid out of an investment contract under which the annuitant(s) deposit certain sums (in a lump sum or in installments) with an annuity guarantor (usually a government agency or an insurance company). The amount paid back includes principal and interest.

1.2.1 Terminology of Annuity Four parties to an annuity :

It Annuitant : A person who receives an annuity is called the annuitant.

Issuer : A company (usually an insurance company) that issues an annuity.

Owner : An individual or an entity that buys an annuity from the issuer of the annuity and makes contributions to the annuity.

Beneficiary : A person who receives a death benefit from an annuity at the death of the annuitant.

Two phases of an annuity

1) **Accumulation phase :** The accumulation (or investment) phase is the time period when money is added to the annuity. An annuity can be purchased in one

single lump sum (known as a single premium annuity) or by making investments periodically over time.

2) **Distribution phase :** The distribution phase is when the annuitant receiving distributions from the annuity. There are two options for receiving distributions from an annuity. The first option is to withdraw some or all of the money in the annuity in lump sums. The second option (commonly known as guaranteed income or annuitization option) provides a guaranteed income for a specific period of time or the entire lifetime of the annuitant.it is said to be a

Types of Annuities :

There are three types of annuities.

1) **Annuity Certain :** An annuity certain is an investment that provides a series of payments for a set period of time to a person or to the person's beneficiary. It is an investment in retirement income offered by insurance companies. The annuity may also be taken as a lump sum. Because it has a set expiration date, a annuity certain generally pays a higher rate of return than lifetime annuity. Typical terms are 10, 15, or 20 years.

2) **Contingent Annuity :** Contingent annuity is a form of annuity contract that provides payments at the time when the named contingency occurs.

   For instance, upon death of one spouse, the surviving spouse will begin to receive monthly payments. In a contingent annuity policy the payment will not be made to the annuitant or the beneficiary until a certain stated event occurs.

3) **Perpetual Annuity or Perpetuit :** A perpetual annuity, also called a perpetuity promises to pay a certain amount of money to its owner forever. Though a perpetuity may promise to pay you forever, its value isn't infinite. The bulk of the value of a perpetuity comes from the payments that you receive in the near future, rather than those you might receive 100 or even 200 years from now.

4) **Classification of Annuities :** Annuities are classified in three categories according to the commencement of income. These three categories are: Immediate Annuity, Annuity Due, and Deferred Annuity.

5) **Immediate Annuity or Ordinary Annuity :** The immediate annuity commences immediately after the end of the first income period. For instance, if the annuity is to be paid annually, then the first installment will be paid at the expiry of one year. Similarly in a half- yearly annuity, the payment will begin at the end of six months. The annuity can be paid either yearly,half-yearly, quarterly or monthly. The purchase money (or consideration) is in a single amount. Evidence of age is always asked for at the time of entry.

6) **Annuity Due :** Under this annuity, the payment of installment starts from the time of contract. The first payment is made as soon as the contract is finalized. The premium is generally paid in single amount but can be paid in installments as is discussed in the deferred annuity. The difference between the annuity due and immediate annuity is that the payment for each period is paid in its beginning under the annuity due contract while at the end of the period in the immediate annuity contract.

7) **Deferred Annuity :** In this annuity contract the payment of annuity starts after a deferment period or at the attainment by the annuitant of 23 a specified age. The premium may be paid as a single premium or in installments.

8) **Present value of an annuity :** The present value of an annuity is the current value of future payments from an annuity, given a specified rate of return or discount rate. The annuity's future cash flows are discounted at the discount rate. Thus the higher the discount rate, the lower the present value of the annuity.

9) **Future value of an annuity :** The future value of an annuity represents the amount of money that will be accumulated by making consistent investments over a set period, assuming compound interest. Rather than planning for a guaranteed amount of income in the future by calculating how much must be invested now, this formula estimates the growth of savings given a fixed rate of investment for a given amount of time.

   The present value of an annuity is the sum that must be invested now to guarantee a desired payment in the future, while the future value of an annuity is the amount to which current investments will grow over time.

10) **Immediate Annuity :** Payments are made at the end of every time period in immediate annuity. Basic formula for an immediate annuity - The accumulated value A of an immediate annuity for n annual payments of an amount.

   C at an interest rate r per cent per annum, compounded annually, is given by.

   $A = \frac{C}{i}\ [(1 + i)^n - 1]$ where $i = \frac{r}{100}$

   Also, the present value P of such an immediate annuity is given by

   $P = \frac{C}{i}\ [1 - (1 + i)^n]$ where $i = \frac{r}{100}$

   Also A = $P\ (1 + \overline{C})\ n$

   $\frac{1}{P} - \frac{1}{A} = \frac{i}{C}$

   Annuity Due Payments are made at the beginning of every time period in annuity due. Basic formula for an annuity due Let C denote the amount paid at the beginning of each of n years and let r denote the rate of interest per cent per annum.

Let $i = \frac{r}{100}$

The accumulated value A' is given by

$$A' = \frac{C\ (1 + i)}{i}\ [(1 + i)^n - 1]$$

The present value P' is given by

$$p^1 = \frac{C\ (1 + i)}{i}\ [1 - (1 + i)^n]$$

A' and P' have the following relations.

$$A' = P'\ (1 + i)^n$$

$$\frac{1}{P^1} - \frac{1}{A^1} = \frac{i}{C\ (1 + i)}$$

Solved Problems :

Ex. 1) A building worth Rs. 50,00,000 is insured for $\left(\frac{4}{5}\right)^{th}$ of its value at a premium of 5%. Find the amount of premium. Also, find commission of the agent if the rate of commission is 3%.

Solution : Property value = Rs. 500000

policy value $= \left(\frac{4}{5}\right) \times 5000000$

= Rs. 400000

Rate of premium = 5%

$\therefore$ Amount of premium = $4000000 \times \frac{5}{100}$

$\therefore$ Premium amount = Rs. 200000

$\therefore$ Commission at 3% = 200000 × 3100

= Rs. 6000

$\therefore$ Agent's commission = Rs. 6000

Ex. 2) A shopkeeper insures his shop valued Rs. 20 lakh for 80% of its value. He pays a premium of Rs. 80000. Find the rate of premium. If the agent gets commission at 12%, find the agent's commission.

Solution : Property value = Rs. 2000000

Insured value = 80% of property value

$$= 2000000 \times \frac{50}{100}$$

$$= \text{Rs. } 1600000$$

Now, the premium paid = Rs. 80000

$$\therefore \text{ Rate of premium} = \frac{100 \times 80000}{1600000}$$

$\therefore$ Rate of premium = 5%

$$\text{Commission paid at 12\% of premium} = 80000 \times \frac{12}{100}$$

$$= \text{Rs. } 9600.$$

$\therefore$ Agent's commission is Rs. 9600.

Ex. 3) A car worth Rs. 5,40,000 is insured for Rs. 4,50,000. The car is damaged to the extent of Rs. 2,40,000 is an accident. Find the amount of compensation that can be claimed under the policy.

Solution : Value of the car = Rs. 540000

Insured value = Rs. 450000 and Damage = Rs. 240000

$$\text{Claim} = \text{Loss} \times \frac{\text{Policy Value}}{\text{Property Value}} = \frac{450000}{540000} \times 240000$$

$$= \text{Rs. } 200000$$

$\therefore$ A compensation of Rs. 2,00,000 can be claimed under the policy.

Ex. 4) 10000 copies of a book, priced Rs. 80 each were insured for th of their value. Some copies of the book were damaged in transit, and were therefore reduced to 60% of their value. If the amount recovered against the damage was Rs.24000, find the number of damaged copies of the book.

Solution : Total number of copies = Rs. 10000 , Cost of one book = Rs.80

$$\text{Insured value} = \frac{3}{5} \times \text{Property value}$$

Insurance claim = Rs. 24000

$$\text{Now, Claim} = \text{Loss} \times \frac{\text{Policy Value}}{\text{Property Value}}$$

$$\therefore 24000 = \frac{3}{5} \times \text{Loss}$$

$\therefore$ Loss $= 24000 \times \frac{3}{5}$

$= \text{Rs. } 40000$

This amount was equal to 40% of the damage.

$\therefore$ Total damage $= 40000 \times \frac{100}{80}$

$= \text{Rs. } 100000$

Since cost of one book was Rs. 80

The number of books damaged $= \frac{100000}{80}$

$\therefore$ 1250 books were damaged.

Ex. 5) A cargo valued at Rs.10,00,000 was insured for Rs.7,00,000 during a voyage. If the rate of premium is 0.4%

find (i) the amount of premium, (ii) The amount that can be claimed if the cargo worth Rs.6,00,000 is destroyed, (iii) the amount that can be claimed, if cargo worth Rs.6,00,000 is destroyed completely and the remaining cargo is so damaged that its value is reduced by 40%.

Solution : Property value = Rs. 10,00,000

Policy value = Rs. 7,00,000

Rate of premium = 0.4%

i) Premium = 0.4% of policy value $= 700000 \times \frac{0.4}{100}$

$= \text{Rs. } 2800$

$\therefore$ 100 Premium = Rs. 2800

ii) Claim $= \text{loss} \times \frac{\text{Policy Value}}{\text{Property Value}}$

$= 600000 \times \frac{700000}{1000000}$

$= \text{Rs. } 420000$

iii) Total value of cargo = Rs. 1000000

Value of the cargo completely destroyed = Rs. 600000

$\therefore$ Value of remaining cargo = Rs. 400000

Loss on value of remaining cargo = 40% of the value of remaining cargo

$$= \frac{0.4}{100} \times 400000$$

= Rs. 160000

$\therefore$ Total Loss = 600000 + 160000 = Rs.760000

$$\text{Claim} = \text{loss} \times \frac{\text{Policy Value}}{\text{Property Value}}$$

$$= 760000 \times \frac{700000}{1000000}$$

= Rs. 532000

Ex. 6) Find the accumulated value after 3 years of an immediate annuity of Rs. 5000 p.a. with interest compounded at 4% p.a. [given $(1.04)^3 = 1.12490$]

Solution : The problem states that $C = 5000$, $r = 4\%$ p.a., and $n = 3$ years.

Then, accumulated value is given by

$$A = \frac{C}{i}[(1 + i) - 1] \text{ where } i = \frac{r}{100} = 0.04$$

$$= \frac{5000}{0.04}[(1 + 0.04)^3 - 1] = \frac{5000}{0.04}[(1.04)^3 - 1] = \frac{5000}{0.04}[1.1249 - 1]$$

$$\therefore A = \frac{5000}{0.04}[1.1249 - 1] = A = \frac{5000}{0.04} \times 0.1249$$

$\therefore A = 5000 \times 3.1225 = 12612.50$

The accumulated value of the annuity is Rs. 15,612.50

Ex. 7) A person plans to accumulate a sum of Rs. 5,00,000 in 5 years for higher education of his son. How much should he save every year if he gets interest compounded at 10% p.a.? [Given $(1.10)^5 = 1.61051$]

Solution : From the problem,

we have A = Rs. 5,00,000, $r = 10\%$ p.a., and $n = 5$ years.

$$\therefore i = \frac{r}{100} = \frac{10}{100} = 0.10$$

We find C as follows,

$$A = \frac{C}{i}[(1 + i)^n - 1] \text{ where i} = \frac{10}{100} = 0.10$$

$\therefore\ 500000 = \dfrac{C}{0.1}\ [(1 + 0.1)^5 - 1]$

$\therefore\ 500000 = \dfrac{C}{0.1}\ [(1.1)^5 - 1]$

$\therefore\ 500000 = \dfrac{C}{0.1}\ [1.61051 - 1]$

$\therefore\ 500000 = \dfrac{C}{0.1}\ (0.61051)$

$\therefore\ C = \dfrac{500000 \times 0.1}{0.6105}$

$\therefore\ C = \dfrac{500000}{0.6105} = 81898.74$

That is, the person should save Rs. 81898.74 every year for 5 years to get Rs. 5,00,000 at the end of 5 years.

Ex. 8) Mr. X saved Rs. 5000 every year for some years. At the end of this period, he received an accumulated amount of Rs. 23205. Find the number of years if the interest was compounded at 10% p.a.[Given $(1.1)^4 = 1.4641$]

Solution : From the problem, we have

A = Rs. 23205, C = Rs. 5000 and r = 10% p.a.

$\therefore\ i = \dfrac{r}{100} = \dfrac{10}{100} = 0.10$

The value of n is found as follows.

$A = \dfrac{C}{i}\ [(1 + i)^n - 1]$ where $= i\ \dfrac{10}{100} = 0.10$

$\therefore\ 23205 = \dfrac{5000}{0.0} = [(1 + 0.1)^n - 1]$

$\therefore\ \dfrac{23205 \times 0.1}{5000} = [(1.1)^n - 1]$

$\therefore\ \dfrac{2320.5}{5000} = [(1.1)^n - 1]$

$\therefore\ 0.4641 + 1 = (1.1)^n\ \therefore\ 1.4641 = (1.1)^n$

since $(1.1)^4 = 1.4641\ \therefore\ n = 4$ years

Ex. 9) Find the rate of interest compounded annually if an immediate annuity of Rs. 20,000 per year amounts to Rs. 41,000 in 2 years.

Solution : From the problem, we have

A = Rs. 41,000, C = Rs. 20,000 and $n = 2$ years.

The value of n is then found as follows.

$$A = \frac{C}{i}\left[(1+i)^n - 1\right]$$

$$41000 = \frac{20000}{i}\left[(1+i)^2 - 1\right]$$

$$\therefore \frac{41000}{20000} = \frac{(1+i)^2 - 1}{i} \quad \therefore \frac{41000}{20000} = \frac{1 + 2i + i^2 - 1}{i}$$

$$\therefore 2.05 = \frac{2i + i^2}{i} => 2 + i$$

$$\therefore i = 05 \text{ but } i = \frac{r}{100} \therefore r = i \times 100 \therefore r = 5$$

$\therefore$ the rate of interest is 5% p.a.

I) Multiple choice questions :

1) Following are different types of insurance.

I) Life insurance　　II) Health insurance　　III) Liability insurance

a) Only I　　b) Only II

c) Only III　　d) All the three

2) By taking insurance, an individual

a) Reduces the risk of an accident

b) Reduces the cost of an accident

c) Transfers the risk to someone else

d) Converts the possibility of large loss to certainty of a small one

3) Amount of money today which is equal to series of payments in future is called

a) Normal value of annuity　　b) Sinking value of annuity

c) Present value of annuity　　d) Future value of annuity

4) Rental payment for an apartment is an example of

a) Annuity due　　b) Perpetuity

c) Ordinary annuity　　d) Installment

5) A retirement annuity is particularly attractive to someone who has

a) A severe illness

b) Risk of low longevity

c) Large family

d) Chance of high longevity

II) Fill in blanks :

1) An installment of money paid for insurance is called ....

2) The value of property is called ....

3) The payment of each single annuity is called ....

4) An annuity where payments continue forever is called ....

5) If payments of an annuity fall due at the end of every period, the series is called annuity ....

III) State whether each of the following is True or False :

1) Accident insurance has a period of five years.

2) Payment of every annuity is called an installment.

3) The present value of an annuity is the sum of the present value of all installments.

4) The future value of an annuity is the accumulated values of all installments.

5) Sinking fund is set aside at the beginning of a business.

IV) Solve the following problems :

1) Find the premium on a property worth Rs. 25,00,000 at 3% if (i) the property is fully insured, (ii) the property is insured for 80% of its value.

2) A building is insured for 75% of its value. The annual premium at 0.70 per cent amounts to Rs. 2625. If the building is damaged to the extent of 60% due to fire, how much can be claimed under the policy?

3) A car valued at Rs. 8,00,000 is insured for Rs. 5,00,000. The rate of premium is 5% less 20%. How much will the owner bear including the premium if value of the car is reduced to 60 % of its original value.

4) A person invested Rs. 5,000 every year in finance company that offered him interest compounded at 10% p.a., what is the amount accumulated after 4 years? [Given $(1.1)^4 = 1.4641$]

5) Find the present value of an annuity immediate of Rs. 36,000 p.a. for 3 years at 9% p.a. compounded annually. [Given $(1.09)^{-3} = 0.7722$]

6) Find the accumulated value of annuity due of Rs. 1000 p.a. for 3 years at 10% p.a. compounded annually. [Given $(1.1)^3 = 1.331$]

7) For an annuity immediate paid for 3 years with interest compounded at 10% p.a., the present value is Rs. 24,000. What will be the accumulated value after 3 years? [Given $(1.1)^3 = 1.331$]

V) Activity based problems :

1) Property Value = Rs. 1,00,000

Policy value = 70% of property value = $\boxed{}$

Rate of premium = 0.4%

Amount of premium = $\frac{0.4}{100} \times \boxed{}$ = Rs. 280

Property worth Rs. 60,000 is destroyed

$$\therefore \text{ Claim} = \text{loss} \times \frac{\text{Policy Value}}{\text{Property Value}}$$

$$= \boxed{} \times \frac{70000}{\boxed{}}$$

= Rs. 42,000

Now, the property worth Rs. 60,000 is totally destroyed and in addition the remaining property is so damaged as to reduce its value by 40%

$$\therefore \text{ Loss} = 60{,}000 + \frac{40}{100} \times \boxed{}$$

= 60,000 + 16,000

= Rs. $\boxed{}$

$$\therefore \text{ claim} = \boxed{} \times \frac{70000}{10000} = \text{Rs. } 53200$$

2) For an immediate annuity, P = Rs. 2000, A = Rs. 4000 and r = 10% p.a.

$$\therefore i = \frac{r}{100} = \frac{\boxed{}}{100} = 0.1$$

$$\therefore \frac{1}{P} - \frac{1}{A} = \frac{1}{C}$$

$$\therefore \frac{1}{\boxed{}} - \frac{1}{\boxed{}} = \frac{0.1}{C}$$

$$\therefore \frac{\boxed{}}{4000} = \frac{0.1}{C}$$

$$\therefore \quad C = \text{Rs. } \boxed{}$$

3) The cost of machinery = Rs. 10,00,000

Effective life of machinery = 12 years

Scrap value of machinery = Rs. 50,000 and r = 5% p.a.

$$\therefore i = \frac{r}{100} = \frac{\boxed{}}{\boxed{}} = 0.05$$

$$\therefore = 10,00,000 - 50,000 = \boxed{}$$

For an immediate annuity

$$A = \frac{C}{i}[(1+i)^n - 1] \text{ where } i = \frac{r}{100}$$

$$\therefore \boxed{} = \frac{C}{\boxed{}}[1 + 0.5)^{\boxed{}} - 1]$$

$$\therefore 950000 = \frac{C}{0.05}\left(1.797 - \boxed{}\right)$$

$$C = \frac{950000 \times \boxed{}}{0.797} \qquad = \text{Rs. } 59598.40$$

■■■

3. Linear regression

3.1.2 Introduction :

Correlation coefficient measures the extent of association between two variables but cannot determine the value of one variable when the value of another variable is known or given.

When two or more variables are correlated, a statistical method to formulate an algebraic relationship between two or more variables in the form of equation to estimate the value of one variable when the values of all other variables are known or specified, is called as **Regression Analysis.**

The variable whose value is predicted is called response or **dependent variable.** The variables used for prediction the response variable or dependent variable is called predictors or **independent variables.**

Linear regression suggests that the relationship between two or more variables is explained by a linear equation, which is known as linear regression model. The linear regression model consists of unknown coefficients. These unknown coefficients are called parameters of the linear regression model.

Correlation Analysis measures the strength and direction of relationship.

Regression Analysis model goes beyond this and develops a formula for predicting the value of response or dependent variable when the values of predictor or independent variables are known.

Types of Linear Regression : Regression equation is the mathematical equation that provides prediction of values of the dependent variable based on the known or given values of the independent variables.

When the linear regression model represents the relationship between the dependent variable and only one independent variable, then the corresponding regression is called a simple linear regression model.

When the linear regression model represents the relationship between the dependent variable and two or more independent variables, then the corresponding regression model is called a multiple linear regression model.

3.2.1. Method of Least Squares :

Based on this principle of method of least squares, the line of best fit are defined for the prediction of value.

When observations on two variables X and Y are available, it is possible to fit a linear regression of Y on X as well as linear regression of X on Y. The models are given separately to understand the difference as well as relationship between the two variables.

3.2.2. Regression of *Y* on *X* :

Linear regression of Y on X assumes that the variable X is independent and Y is dependent variable. To make it explicit, the linear regression model for Y on X is given as follows. $\overline{X}$ is mean of variable X and $\overline{Y}$ is mean of variable Y.

$Y - \overline{Y} = b_{XY} (X - \overline{X})$ where b_{YX} is regression coefficient of Y on X.

$$b_{yx} = \frac{\text{Covariance } (X, Y)}{\text{Variance } (X)}$$

$$b_{yx} = \frac{\Sigma (X - \overline{X}) (Y - \overline{Y})}{\Sigma (X - \overline{X})^2} = \frac{\Sigma XY - n\overline{X}\ \overline{Y}}{\Sigma X^2 - n (\overline{X})^2}$$

3.2.3. Regression of X on Y :

Linear regression of X on Y assumes that the variable Y is independent and X is dependent variable. To make it explicit, the linear regression model for X on Y is given as follows.

$X - \overline{X} = b_{XY} (Y - \overline{Y})$ where b_{XY} is regression coefficient of X on Y.

$$b_{XY} = \frac{\text{Covariance } (X, Y)}{\text{Variance } (Y)}$$

$$b_{XY} = \frac{\Sigma (X - \overline{X}) (Y - \overline{Y})}{\Sigma (Y - \overline{Y})^2} = \frac{\Sigma XY - n\overline{X}\ \overline{Y}}{\Sigma Y^2 - n (\overline{Y})^2}$$

Note that the point ($\overline{X}, \overline{Y}$) satisfies both the lines of regression. Hence it is a point of intersection of two lines of regression.

3.3. Properties of Regression Coefficient :

The slope of regression line of Y on X (b_{YX}),

i.e. $Y = a + b_{YX} X$, is called as the regression coefficient of Y on X.

Similarly, the slope of regression line of X on Y (b_{XY}),

i.e. $X = a' + b_{xy} Y$, is called as the regression coefficient of X on Y.

1) $b_{YX} \cdot b_{XY} = r^2$ or $r = \sqrt{b_{YX} \cdot b_{XY}}$

   where 'r' is the correlation coefficient between X and Y.

   $\therefore$ The sign of correlation coefficient 'r' is same as the sign of both b_{YX} and b_{XY}.

2) If $b_{YX} > 1$, then $b_{XY} < 1$.

   $b_{YX} = r. \frac{\sigma_y}{\sigma_x}$ and $b_{XY} = r. \frac{\sigma_x}{\sigma_y}$ where σ_x and σ_y is standard deviation of X and Y respectively.

3) $\left|\dfrac{b_{XY} + b_{YX}}{2}\right| \geq |r|$

4) b_{YX} and b_{XY} are not affected by change of origin, but are affected by change of scale.

Solved Examples :

Ex. 1) For a certain bivariate data on 5 pairs of observations given.

$\Sigma X = 20$, $\Sigma Y = 20$, $\Sigma X^2 = 90$, $\Sigma Y^2 = 90$, $\Sigma XY = 76$

Calculate i) Cov (*X*, *Y*). ii) b_{yx} and b_{xy} iii) *r*

Solution : $n = 5$, $\overline{X} = \dfrac{\Sigma X}{n} = 4$, $\overline{X} = \dfrac{\Sigma Y}{n} = 4.$

i) Covariance $(X, Y) = \dfrac{\Sigma XY - n\overline{X}\ \overline{Y}}{5}$

$= \dfrac{76 - 5(4)(4)}{5}$

$= \dfrac{-4}{5} = -0.8$

ii) $b_{YX} = \dfrac{\Sigma XY - n\overline{X}\ \overline{Y}}{\Sigma X^2 - n(\overline{X})^2}$

$= \dfrac{76 - 5(4)(4)}{90 - 5(4)(4)}$

$b_{YX} = \dfrac{-2}{5} = -0.4$

Now $b_{XY} = \dfrac{\Sigma XY - n\overline{X}\ \overline{Y}}{\Sigma X^2 - n(\overline{X})^2}$

$= \dfrac{76 - 5(4)(4)}{90 - 5(4)(4)}$

$b_{XY} = \dfrac{-2}{5} = -0.4$

iii) $r^2 = b_{YX} \cdot b_{XY}$

$= (-0.4) \cdot (-0.4)$

$\therefore r^2 = 0.16$

Since both b_{YX} and b_{XY} are negative.

$\therefore r = -0.4.$

Ex. 2) From the following data, obtain two regression equations.

i) Estimate Y when $X = 5$, ii) find X when $Y = 10$.

Independent Variable (X)	4	6	8
Independent Variable (Y)	12	8	7

Solution :

X	Y	$X - \overline{X}$	$Y - \overline{Y}$	$(X - \overline{X})^2$	$(Y - \overline{Y})^2$	$(X - \overline{X})(Y - \overline{Y})$
4	12	–2	3	4	9	– 6
6	8	0	–1	0	1	0
8	7	2	–2	4	4	– 4
18	27	0	0	8	14	–10

$n = 3, \Sigma X = 18, \Sigma Y = 27$

$\overline{X} = \frac{\Sigma X}{n} = 6, \overline{Y} = \frac{\Sigma Y}{n} = 9$

The regression coefficient of Y on X is b_{YX},

$$b_{YX} = \frac{\text{Covariance } (X, Y)}{\text{Variance } (X)}$$

$$\text{Covariance } (X, Y) = \frac{\Sigma (X - \overline{X})(Y - \overline{Y})}{n} = \frac{-10}{3}$$

$$\text{Var } (X) = \frac{\Sigma (X - \overline{X})^2}{n} = \frac{8}{3}$$

$$b_{YX} = \frac{-10}{8} = \frac{-5}{4}$$

Here the line of regression of Y on X is

$y - \overline{Y} = b_{YX} (x - \overline{X})$

$\therefore y - 9 = \frac{-5}{4} (x - 6)$

$\therefore 5x + 4y = 66$ or $y = -1.2x + 16.5$

Now, The regression coefficient of X on Y is b_{XY}

$$b_{xy} = \frac{\text{Covariance } (X, Y)}{\text{Variance } (Y)}$$

$$\text{Var } (Y) = \frac{\Sigma (Y - \overline{Y})^2}{n} = \frac{14}{3}$$

$$\therefore b_{XY} = \frac{-10}{14} = \frac{-5}{7}$$

Here the line of regression of X on Y is

$X - \overline{X} = b_{XY} (Y - \overline{Y})$

$\therefore X - 6 = \frac{-5}{7} (Y - 9)$

$\therefore 7X + 5Y = 87$ or $X = -0.71Y + 12.43$

i) Estimate of Y when $X = 5$ is

$Y = -1.2 (5) + 16.5$

$= 10.5$

ii) Estimate of X when $Y = 10$ is

$X = -0.71 (10) + 12.43$

$= 5.33$

Ex. 3) Given: $n = 8$, $\Sigma (X - \overline{X})^2 = 36$, $\Sigma (Y - \overline{Y})^2 = 40$, $\Sigma (X - \overline{X})(Y - \overline{Y}) = 24$, $\overline{X} = 13$, $\overline{Y} = 17$. Obtain the two regression equations.

Solution : $b_{YX} = \frac{\Sigma (X - \overline{X})(Y - \overline{Y})}{\Sigma (X - \overline{X})^2}$

$b_{YX} = \frac{24}{36} = \frac{2}{3}$

$\therefore b_{XY} = \frac{\Sigma (X - \overline{X})(Y - \overline{Y})}{\Sigma (Y - \overline{Y})^2}$

$= \frac{24}{40} = \frac{3}{5}$

$\therefore$ Regression equation of Y on X is

$\therefore Y - \overline{Y} = b_{YX} (X - \overline{X})$

$\therefore Y - 17 = \frac{3}{5} (X - 13)$

$\therefore Y = 0.66 X - 8.33$ or $3Y = 2X + 25$

$\therefore$ Regression equation of X on Y is

$\therefore X - \overline{X} = b_{XY} (Y - \overline{Y})$.

$\therefore X - 13 = \frac{3}{5} (Y - 17)$

$\therefore X = 0.6 Y + 2.8$ or $5X = 3Y + 14$

Ex. 4) If the regression lines of Y on X and X on Y are $2x - y - 1 = 0$ and $7x - 2y = 8$ respectively, then find (i) $\overline{X}$ and $\overline{Y}$ (ii) b_{YX} and b_{XY} (iii) r (iv) if Var (X) = 4, obtain Var (Y).

Solution : i) We know that $(\overline{X}, \overline{Y})$ is, a point of intersection of two lines of regression.

$\therefore$ Solve the equations (1) and (2) simultaneously.

$4x - 2y = 2$........................(1)

$7x - 2y = 8$........................(2)

(−). (+). (−)

$3x = 6$

i.e. $x = 2$

Substitute $x = 2$ in one of the equations, we get y = 3

$\therefore \overline{X} = 2, \ \overline{Y} = 3$

ii) We know that slope of regression line is the regression coefficient.

$\therefore$ the line of Y on X, i.e. $Y = 2X - 1$ gives $b_{YX} = 2$

Similarly, the line of X on Y, $7X - 2Y = 8$ gives

$X = \frac{2}{7} Y + \frac{8}{7}$ i.e $b_{XY} = \frac{2}{7}$

$\therefore b_{YX} = 2$ and $b_{XY} = \frac{2}{7}$

iii) Now to find 'r',

$r^2 = b_{YX} * b_{XY}$

$= 2 * \frac{2}{7}$

$\therefore r = \sqrt{\frac{4}{7}} = \sqrt{0.5714} = 0.76$

iv) To find Var (Y) when Var (X) = 4

$\therefore$ S.D (X) = $\sigma x = 2$

$\therefore b_{YX} = r.\frac{\sigma_y}{\sigma_x}$

$\therefore 2 = \sqrt{\frac{4}{7}} . \frac{\sigma_y}{2}$

$\therefore \sigma y = 2\sqrt{7} = \sqrt{28}$

$\therefore$ Var (Y) = 28

Ex. 5) Given the following information about the demand and supply of a commodity, obtain two regression lines.

	Demand (in tons) (X)	**Demand (in tons) (Y)**
Mean	13	17
Std. Deviation	3	2

Correlation coefficient between demand and Supply is 0.6. Estimate supply when demand is 10 tons and estimate demand when supply is 15 tons.

Solution : Given that $\overline{X} = 13$, $\overline{Y} = 17$, $\sigma_x = 3$, $\sigma_y = 2$, $r = 0.6$

Then to find b_{YX} and b_{XY}

$$b_{YX} = \underline{r}. \frac{\sigma_y}{\sigma_x} = 0.6 \; . \; \frac{2}{3} = 0.4$$

$$b_{YXX} = \underline{r}. \frac{\sigma_x}{\sigma_y} = 0.6 \; . \; \frac{3}{2} = 0.9$$

$\therefore$ Regression equation of Y on X is

$\therefore \; Y - \overline{Y} = b_{YX} (X - \overline{X})$

$\therefore \; Y - 17 = 0.4 (X - 13)$

$\therefore \; Y = 0.4 X + 11.8$

When $X = 10$, $\therefore$ Y = 0.4 (10) + 11.8

$\therefore \; Y = 15.8$ tons

Similarly, regression line of X on Y is

$\therefore \; X - \overline{X} = b_{XY} (Y - \overline{Y})$.

$\therefore \; X - 13 = 0.9 (Y - 17)$.

$\therefore \; X = 0.9 \; Y - 2.3$.

When $Y = 15$, $\therefore \; X = 0.9 (15) - 2.3$

$\therefore \; X = 11.2$ tons.

Miscellaneous Exercise

A) Choose the correct alternative for each of the following.

1) In Regression analysis, the variable whose value is to be predicted is called as ....

a) Predictor   b) Regressor   c) Response   d) Independent

2) The linear regression model can be fitted in .... way.

a) 4   b) 3   c) 2   d) 1

3) If Covariance $(x , y) > 0$, then b_{YX} is ....

a) Negative. b) positive c) zero d) 1

4) If Covariance $(x , y) < 0$, then b_{YX} = ...

a) 1 b) positive c) Negative d) zero

5) If $b_{YX} = -0.2$ and $r = -0.6$, then b_{XY} is

a) 1.8 b) − 0.3. c) − 1.8. d) 0.3

6) If $b_{XY} = \frac{1}{6}, r^2 = \frac{1}{4}$, then b_{YX} = ...

a) $\frac{3}{2}$ b) $-\frac{3}{2}$ c) $\frac{2}{3}$ d) $-\frac{2}{3}$

7) If $b_{XY} < 1$ then b_{YX} is ....

a) Greater than 1 b) less than 1

c) only (b) d) both (a) and (b).

8) One can predict or estimate the value of one variable with the help of another variable only if they are ....

a) Uncorelated b) positively correlated.

c) correlated. d) negatively correlated

9) For a bivariate data, the regression coefficient of Y on X is 0.4 and regression coefficient of X on Y is 0.9 then the correlation coefficient between X and Y is ....

a) 1.3 b) 0.6 c) 0.5 d) 0.36.

10) If the regression equation of X on Y is $5x - 2y = 40$ and mean value of Y is 5, then mean value of X is ....

a) 12.5. b) 10 c) 8.5 d) 5

11) For a bivariate data, the standard deviation of x and y is 3.5 and 28 respectively. If the coefficient of correlation between x and y is 0.8 then b_{XY} is ....

a) 0.1 b) 0.2 c) 0.64 d) 1

12) For a bivariate data, the standard deviation of x and y is 3.5 and 28 respectively. If the coefficient of correlation between x and y is 0.8 then b_{YX} is ....

a) 0.64 b) 2 c) 6.4 d) 1.2

13) If $b_{YX} = 1.7$ then the possible value of b_{XY} will be....................

a) 1.9 b) 1 c) 0.8. d) 0.3

14) $b_{YX} \cdot \sigma x =$ ....

a) Cov $(x, y) \cdot \sigma y$

b) Cov $(x, y) . \sigma x$

c) $r . \sigma x$

d) $r . \sigma y$

15) If r is negative, then b_{YX} and b_{XY} are ....

a) Positive b) negative c) zero d) insufficient data

B) Fill in the blanks.

1) If $b_{YX} = -0.9$, $r = -0.6$, then b_{XY} is ....
2) Correlation between Y ans $-Y$ is ....
3) $|b_{XY} + b_{YX}| \geq$ ....
4) If $u = x - a$ and $v = y - b$ then $b_{XY} =$ ....
5) If $r = 0.5$, $\sigma x = 4$ and $\sigma y = 5$ then b_{XY} is ....
6) If r = 0.5, $\sigma x = 4$ and $\sigma y = 5$ then b_{YX} is ....
7) If r = 0.6 and $b_{YX} = 0.5$ then $\frac{\sigma_x}{\sigma_y}$ is ....
8) If the regression equation of X on Y is $5x = 2y + 40$ and mean value of X is 10, then mean value of Y is ....
9) If the regression line of Y on X is $3x + 2y = 26$, then b_{YX} is ....
10) If the regression line of X on Y is $5x - 7y = 12$, then b_{XY} is ....
11) The regression equations of two regression lines are $3x + 2y = 26$ and $5x + 7y = 12$ then the r^2, is ....
12) The regression equation of Y on X and X on Y is $2y - x - 50 = 0$ and $3y - 2x = 10$ respectively, then the likely value of Y when $X = 10$ is ....
13) The regression equation of Y on X and X on Y is $2y - x - 50 = 0$ and $3y - 2x = 10$ respectively, then the likely value of X when $Y = 12$ is ....
14) If $b_{XY} = \frac{2}{5}$ and $\sigma y = 6$ then Cov $(X, Y) =$ ....
15) If $b_{YX} = \frac{9}{10}$ and Cov $(X, Y) = \frac{729}{10}$, then σx is ....

C) State whether each of the following is true or False.

1) In the regression equation of X on Y, b_{YX} is the slope of the regression line.
2) $|b_{XY} + b_{YX}| \leq 2\,|r|$

3) If $u = x - a$ and $v = y - b$ then $b_{YX} = b_{VU}$

4) For a bivariate data, $b_{XY} = -0.3$ and $b_{YX} = -1.2$, then $r = 0.6$.

5) If $r = 0.6$, $\sigma x = 4$ and $\sigma y = 5$ then b_{XY} is 0.48

6) If equations of regression lines are $3x + 2y - 26 = 0$ and $6x + y - 31 + 0$ then the mean of x is 7.

7) If x and y are two variables and the regression equation of X on Y is $x + 2y = 5$. If mean of Y is 2, then mean of x is 1.

8) Given that mean and standard deviation of two variables x and y are 6, 5 and 8, 13 respectively. The value of r is 0.4, then b_{XY} is 0.2.

9) If Cov $(x, y) < 0$, then $b_{YX} > 0$, $b_{XY} > 0$

10) Cov $(x, y) \neq$ Cov (y, x)

11) If $b_{YX} = -1.9$ and $b_{XY} = -0.25$, then $r > 1$

12) If $b_{YX} = 0.45$ and $b_{XY} = 0.05$, then $r = 1.5$

13) If $9x - 4y + 15 = 0$ and $25x - 4y = 17$ are two regression equations, then $9x - 4y + 15 = 0$ is the regression equation of X on Y.

14) If $4x - 5y + 33 = 0$ and $20x - 9y = 107$ are two regression equations, then $4x - 5y + 33 = 0$ is the regression equation of Y on X.

15) Product of b_{YX} and. b_{XY} is always positive.

D) Solve the following problems.

1) For a certain bivariate data on 5 pairs of observations given.

   $\Sigma X = 56$, $\Sigma Y = 56$, $\Sigma X^2 = 476$, $\Sigma Y^2 = 476$, $\Sigma XY = 469$ and $n = 7$.

   a) Find the regression equation of X on Y b) X when Y is 12.

2) From the data of 20 pairs of observations on X and Y, following results are obtained:

   $\Sigma (X - \overline{X})^2 = 1200$, $\Sigma (Y - \overline{Y})^2 = 300$,

   $\Sigma (X - \overline{X})(Y - \overline{Y}) = -250$, $\overline{X} = 200$, $\overline{Y} = 95$

   a) Find the line of regression Y on X b) Estimate Y when X is 210.

3) The manager of a company wants to find a measure which he can use to fix the monthly wages of persons applying for job in the accounts department. As an experimental project, he collected data from 5 persons from the department for the reference.

Years of service (X)	21	25	26	24	19
Income(in 000's) (Y)	19	20	24	21	16

Find the regression equation of income on the years of service. Also find the likely income for a person with 20 years of service.

4) The following table gives the information about the advertising expenditure (in lakh) and Sales (in lakh) of a particular product. Given that the correlation coefficient between the advertising expenditure and Sales is 0.8.

	Advertising expenditure (X)	**Sales (Y)**
Arithmetic Mean	10	90
Standard deviation	3	12

Obtain both the regression equations. Estimate the sales when advertising budget is 12 lakh and likely advertising expenditure when the sales target is 110 lakhs.

5) The regression equation of Y on X is $9y = 2x$ and the regression equation of X on Y is $6x = 3y + 7$. Find (i) correlation coefficient between X and Y (ii) Var (X) if Var (Y) = 4.

6) If for a bivariate data, Mean of $X = 14$, Mean of $Y = 17$, $b_{YX} = -0.6$ and $b_{XY} = -0.216$. Obtain both regression equations. Estimate Y when $X = 20$ and estimate X when $Y = 25$.

7) The equations of two regression lines are $2x + 3y - 5 = 0$ and $5x + 7y - 12 = 0$. Find (a) correlation coefficient, (b)mean values of X and Y.

E) Complete the following activities.

1) The equation of the line of regression of Y on X is $3x + 2y = 26$ and X on Y is $6x + y = 31$. Find (a) the mean of X and Y (b) correlation coefficient (c) Var (x) if Var (y) = 36.

Solution : a) We know that $(\overline{X}, \overline{Y})$ is, a point of intersection of two lines of regression.

$3x + 2y = 26$ .... (i)

$6x + y = 31$.............................. (ii)

Solve equations (i) and (ii), we get

Mean of X = $\boxed{}$, Mean of Y = $\boxed{}$

b) We find the slopes of given regression lines

For $3x + 2y = 26$,

coefficient of X = $\boxed{}$, coefficient of Y = $\boxed{}$,

$$\text{Slope } (m_1) = -\frac{\text{coefficient of } X}{\text{coefficient of } Y} = \frac{\boxed{}}{\boxed{}}$$

Similarly, For $6x + y = 31$, coefficient of $X = 6$, coefficient of $Y = 1$, Slope (m_2) $= -\dfrac{\text{coefficient of } X}{\text{coefficient of } Y} = \dfrac{\square}{\square} = \square$

$\therefore |m_1| < |m_2|$

$\therefore b_{YX} = \square$, $b_{XY} = \square$

$\therefore 3x + 2y = 26$ is the regression equation of $\square$ on $\square$

and $6x + y = 31$ is the regression equation of $\square$ on $\square$

Now, the correlation coefficient $r = \pm \sqrt{b_{YX} \cdot b_{XY}}$

$\therefore r = \pm = \square$

c) If Var (y) = 36, $\sigma y = \sqrt{\square} = \square$

$\therefore b_{YX} = r. \dfrac{\sigma_x}{\sigma_y}$,

$\therefore \sigma_x = \square$

2) From the data of 7 pairs of observations on *X* and *Y* following results are obtained :
$\Sigma (x - 70) = -38$, $\Sigma (y - 60) = -5$, $\Sigma (x - 70)^2 = 2990$, $\Sigma (y - 60)^2 = 475$, $\Sigma (x - 70)(y - 60) = 1063$. Obtain the two regression coefficients.

Solution : Let $u = x - 70$ and $v = y - 60$

$\therefore \Sigma u = -38$, $\Sigma v = -5$, $\Sigma u^2 = 2990$, $\Sigma v^2 = 475$, $\Sigma uv = 1063$.

$\overline{u} = \dfrac{\square}{\square} = \square$ $\quad \overline{v} = \dfrac{\square}{\square} = \square$

$\sigma_u^2 = \dfrac{\Sigma u^2}{n} - (\overline{u}.) = \square$ $\quad \sigma_v^2 = \dfrac{\Sigma v^2}{n} - (\overline{v}.)^2 = \square$

Cov $(u,v) = \dfrac{\Sigma uv^2}{n} - (\overline{u}.\overline{v}) = \dfrac{\square}{\square} = \square$

$b_{YX} = b_{VU} = \dfrac{\text{coefficient of } (U, V)}{\text{coefficient of } (U)} = \dfrac{\square}{\square} = \square$

$b_{XY} = b_{UV} = \dfrac{\text{coefficient of } (U, V)}{\text{coefficient of } (V)} = \dfrac{\square}{\square} = \square$

■■■

4. Time Series

Definition : Time Series is a sequence of observations made on a variable at regular time intervals over a specified period of time.

Some examples from day-to-day life may give a better idea of time series.

1) Monthly, quarterly, or yearly production of an industrial product.
2) Yearly GDP (Gross Domestic Product) of a country.
3) Monthly sales in a departmental store.
4) Weekly prices of vegetables.
5) Daily closing price of a share at a stock exchange.
6) Hourly temperature of a city recorded by the Meteorological Department.

Uses of Time Series Analysis :

The main objective of time series analysis is to understand, interpret and assess chronological changes in values of a variable in the past, so that reliable predictions can be made about its future values.

1) It is useful for studying the past behaviour of a variable.
2) It is useful for forecasting future behaviour of a variable.
3) It is useful in evaluating the performance.
4) It is useful in making a comparative study.

Components of Time Series :

1) **Secular Trend (T) :** The secular trend is the long term pattern of a time series. The secular trend can be positive or negative depending on whether the time series exhibits an increasing long term pattern or a decreasing long term pattern.
2) **Seasonal Variation (S) :** Seasonal variation is the component of a time series that involves patterns of change within a year that repeat from year to year.
3) **Cyclical Variation (C) :** Cyclical variation is a long term oscillatory movement in values of a time series. Cyclical variation occurs over a long period, usually several years. One complete round of oscillation is called a cycle. A typical business cycle consists of the following four phases : (i) prosperity, (ii) recession, (iii) depression, (iv) recovery.
4) **Irregular Variation (I) :** Irregular variations are unexpected variations in time series caused by unforeseen events that can include natural disasters like floods or famines, political events line strikes or agitations, or international events like wars or others conflicts.

Mathematical Models of Time Series

1) **Additive Model :** The additive model assumes that the value X_t at time t is the sum of the four components at time t.

   Thus, $X_t = T_t + S_t + C_t + I_t$

   The additive model assumes that the four components of the time series are independent of one another. It is also important to remember that all the four components in the additive model must be measured in the same unit of measurement.

2) **Multiplicative Model :** The multiplicative model that the value X_t at the time t is obtained by multiplication of the four components at time t.

   That is, $X_t = T_t + S_t + C_t + I_t$

   The multiplicative model does not assume independence of the four components of the series and is, therefore, more realistic. Values of the trend are expressed in units of measurements and other components are expressed as percentage or relative values, and hence are free from units of measurements.

Measurement of Secular Trend

1) **Method of Freehand Curve (Graphical Method) :** In this method, a graph is drawn for the given time series by plotting X_t (on Y-axis) against t (on X-axis). Then a free hand smooth curve is plotted on the same graph to indicate the general trend.

2) **Method of moving Averages :** The moving average of period k of a time series forms a time series of arithmetic means of k successive observations from the original time series. The method begins with the first k observations and finds the arithmetic mean of these k observations. The next step leaves the first observation and includes observation number $k + 1$ and finds the arithmetic mean of these k observations. This process continues till the average of the last k observations is found.

3) **Method of Least Squares :** This is the most objective and perhaps the best method of determining trend in a given time series. The method begins with selection of an appropriate form of trend equation and then proceeds with estimation of the unknown constants in this equation. It is a common practice to choose a polynomial of a suitable degree and then to determine its unknown (but constant) coefficients by the method of least squares. The choice of the degree of polynomial is often based on the graphical representation of the given data.

   In a linear trend, the equation is given by $Xt = a + bt$

   The method of least squares involves solving the following set of linear equations, commonly known as normal equations.

- $\sum Xt = na + b \sum t$

- $\sum tX_t = a\sum t + b\sum t^2$
- Where n is the number of the time periods for which data is available, whereas, $\sum Xt$, $\sum tXt$, $\sum t$ and $\sum t^2$ are obtained from the data. The least squares estimates of a and b are obtained by solving the two equations in the two unknowns, namely a and b. The required equation of the trend line is the obtained by substituting these estimates in equation $Xt = a+bt$.

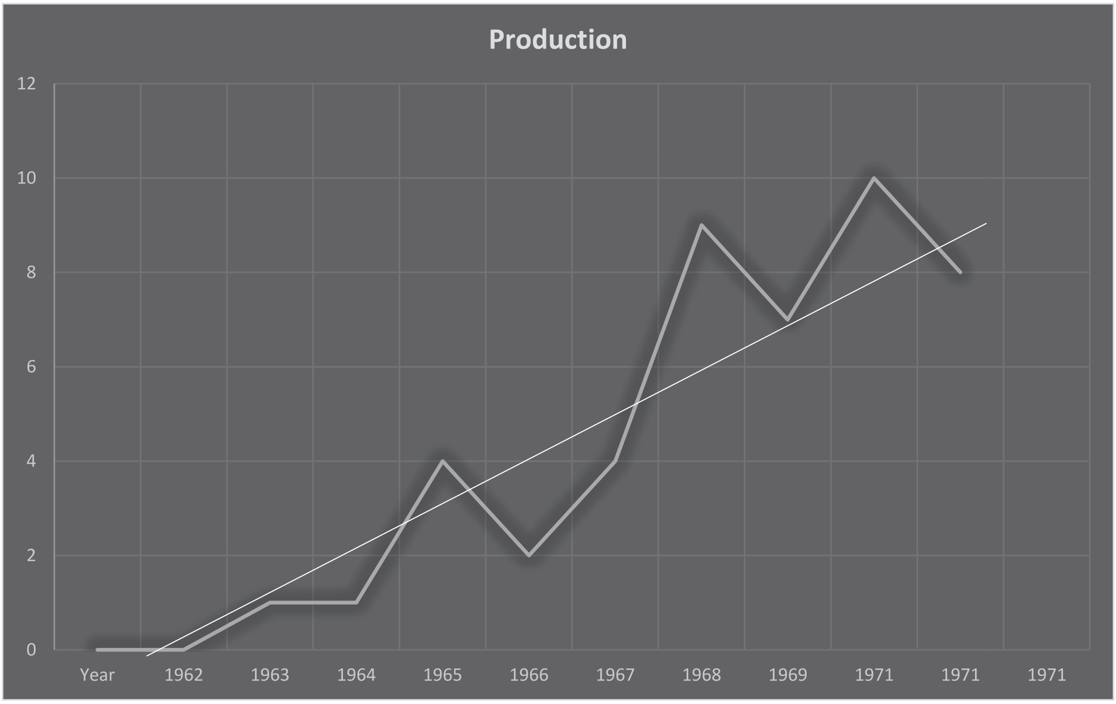
Solved Examples :

1) **The following data gives the production of bleaching powder (in '000 tones) for the years 1962 to 1972.**

Year	1962	1963	1964	1965	1966	1967
Production	0	0	1	1	3	2
Year	1968	1969	1970	1971	1972	
Production	4	6	5	1	4	

Fit a trend line by graphical method to the above data.

Solution :



2) Use the method of least squares to fit a trend line to the data in Problem 1 above. Also, obtain the trend value for the year 1975.

Solution : In the given problem, n = 11 (odd), middle t value is 1967, and h = 1.

We use the transformation $u = \frac{t - 1986}{1} = t - 1967$.

Year (t)	y_t	u	u^2	u y_t
1962	0	–5	25	0
1963	0	–4	16	0
1964	1	–3	9	–3
1965	1	–2	4	–2
1966	4	–1	1	–4
1967	2	0	0	0
1968	4	1	1	4
1969	9	2	4	18
1970	7	3	9	21
1971	10	4	16	40
1972	8	5	25	40
Total	46	0	110	114

Here n = 11,

$\sum y_t = 46$,

$\sum u = 0$, $\sum u^2 = 110$, $\sum u.y_t = 114$

$\therefore$ a $= \frac{\sum y_t}{n} = \frac{46}{11} = 4.182$

And b $= \frac{\sum u y_t}{\sum u^2} = \frac{114}{110} = 1.036$

Equation of trend line is y_t = a + bu

$\therefore$ $Yt = 4.182 + (1.036)\ u$

To estimate trend value for the year 1975, put $u = 8$

$Y_{1975} = 4.182 + 1.036\ (8)$

= 12.47 thousand tones

$\therefore$ Estimated production = 12,470 tones

3) **The following table shows the index of industrial production for the period from 1976 to 1985, using the year 1976 as the base year.**

Year	1976	1977	1978	1979	1980
Index	0	2	3	3	2
Year	1981	1982	1983	1984	1985
Index	4	5	6	7	10

Fit a trend line to the above data by the method of least squares. Also, obtain the trend value for the index of industrial production for the year 1987.

Solution : Let the equation of the trend line be y_t = a + bt

In the given problem,

n=10 (even), two middle t values are 1980 and 1981, and h=1.

We use the transformation $u = \dfrac{t - 1980.5}{\frac{1}{2}} = 2t - 3961.$

The equation of the trend line then becomes y_t = a + bu

$\sum y_t = na + b \sum u$ .... (i)

$\sum uy_t = a\sum u + b\sum u^2$ .... (ii)

Where, $\sum y_t$, $\sum u\ y_t$, $\sum u$ and $\sum u^2$ are obtained from the following table.

Year (t)	y_t	u	u^2	$u\ y_t$
1976	0	–9	81	0
1977	2	–7	49	–14
1978	3	–5	25	–15
1979	3	–3	9	–9
1980	2	–1	1	–2
1981	4	1	1	4
1982	5	3	9	15
1983	6	5	25	30
1984	7	7	49	49
1985	10	9	81	90
Total	42	0	330	148

Here $n = 10$, $\sum y_t = 42$, $\sum u = 0$, $\sum \underline{u}^2 = 330$, $\sum u.y_t = 148$

$\therefore$ a $= \dfrac{\sum y_t}{n} = \dfrac{42}{10} = 4.2$ from Equation (i)

And b $= \dfrac{(\sum u.y_t)}{(\sum u^2)} = \dfrac{148}{330} = 0.4484$ from Equation (i)

$\therefore$ Equation of trend line is y_t = a+bu

$\therefore$ $y_t = 4.2 + (0.4484)\ u$

To estimate trend value for the year 1987, put $u = 13$

$Y_{1987} = 4.2 + (0.4484)\ (13)$

$= 10.0292$

$\therefore$ Estimated index of industrial production = 10.0292

4) Obtain the trend values for Problem 1 using 5 yearly moving averages.

Solution :

Year (t)	**Production**	**5-yearly moving total**	**5-yearly moving averages (trend value)**
1962	0	-	-
1963	0	-	-
1964	1	6	$\frac{6}{5} = 1.2$
1965	1	8	$\frac{8}{5} = 1.6$
1966	4	12	$\frac{12}{5} = 2.4$
1967	2	20	$\frac{20}{5} = 4$
1968	4	26	$\frac{26}{5} = 5.2$
1969	9	32	$\frac{32}{5} = 6.4$
1970	7	36	$\frac{36}{5} = 7.2$
1971	10	-	-
1972	8	-	-

Note : that 5 yearly average is not available for the first 2 years and last 2 years.

5) **Obtain the trend values for the data in problem 3 using 4 - yearly centered moving averages.**

Solution :

Year	Index	4-yearly moving total	4-yearly moving centred total	4-yearly centred moving average (trend value)
1976	0		-	-
		-		
1977	2		-	-
		8		
1978	3		18	$\frac{18}{8} = 2.25$
		10		
1979	3		22	$\frac{22}{8} = 2.75$
		12		
1980	2		26	$\frac{26}{8} = 3.25$
		14		
1981	4		31	$\frac{31}{8} = 3.875$
		17		
1982	5		39	$\frac{39}{8} = 4.875$
		22		
1983	6		50	$\frac{50}{8} = 6.025$
		28		
1984	7		-	-
		-		
1985	10		-	-

I) Choose the correct alternative.

1) Which of the following can't be a component of a time series?

a) Seasonality b) Cyclical c) Trend d) Mean

2) Which component of time series refers to erratic time series movements that follow no recognizable or regular pattern?

a) Trend b) Seasonal c) Cyclical d) Irregular

3) The following trend line equation was developed for annual sales from 1984 to 1990 with 1984 as base or zero year.

$Y = 500 + 60X$ (in 1000 ₹). The estimated sales for 1984 (in 1000 ₹) is :

a) 500 b) 560 c) 1,040 d) 1,100

4) An overall upward or downward pattern in an annual time series would be contained in which component of the times series

a) Trend b) Cyclical c) Irregular d) Seasonal

5) Moving averages are useful in identifying

a) Seasonal component b) Irregular component

c) Trend component d) cyclical component

II) Fill in the blanks.

1) .... components of time series is indicated by a smooth line.

2) .... component of time series is indicated by periodic variation year after year.

3) The complicated but efficient method of measuring trend of time series is ....

4) The simplest method of measuring trend of time series is ....

5) The method of measuring trend of time series using only averages is ....

III) State whether each of the following is True or False.

1) The secular trend component of time series represents irregular variations.

2) Seasonal variation can be observed over several years.

3) Cyclical variation can occur several times in a year.

4) Moving average method of finding trend is very complicated and involves several calculations.

5) Least squares method of finding trend is very simple and does not involve any calculations.

IV) Solve the following problems.

1) Following table shows the amount of sugar production (in lac tons) for the years 1971 to 1982.

Year	1971	1972	1973	1974	1975	1976
Production	1	0	1	2	3	2
Year	1977	1978	1979	1980	1981	1982
Production	4	6	5	1	4	10

Fit a trend line by the method of least squares.

2) Obtain trend values for data in Problem 1 using 4 - yearly centered moving averages.

3) The following table gives the production of steel (in millions of tons) for years 1976 to 1986.

Year	1976	1977	1978	1979	1980	1981
Production	0	4	4	2	6	8
Year	1982	1983	1984	1985	1986	
Production	5	9	4	10	10	

Fit a trend line by the method of least squares. Also, obtain the trend value for the year 1990.

4) Following table shows the amount of sugar production (in lac tons) for the years 1971 to 1982.

Year	1971	1972	1973	1974	1975	1976
Production	1	0	1	2	3	2
Year	1977	1978	1979	1980	1981	1982
Production	4	6	5	1	4	10

Fit a trend line to the above data by graphical method.

5) Use the method of least squares to fit a trend line to the data in Problem 6 below. Also, obtain the trend value for the year 1975.

6) The following table shows the production of gasoline in U.S.A. for the years 1962 to 1976.

Obtain trend values for the above data using 5 - yearly moving averages.

Year	1962	1963	1964	1965	1966	1967	1968	1969
Production (million barrels)	0	0	1	1	2	3	4	5
Year	1970	1971	1972	1973	1974	1975	1976	
Production (million barrels)	6	8	9	9	8	9	10	

7) The following table shows the index of industrial production for the period from 1976 to 1985, using the year 1976 as the base year.

Year	1976	1977	1978	1979	1980
Index	0	2	3	3	2
Year	1981	1982	1983	1984	1985
Index	4	5	6	7	10

Fit a trend line to the above data by graphical method.

V) Activity based questions :

1) Following table shows the all India infant mortality rates (per '000) for years 1980 to 2000.

Year	1980	1985	1990	1995	2000	2005	2010
IMR	10	7	5	4	3	1	0

Fit a trend line by the method of least squares.

Solution : Let us fit equation of trend line for above data.

Let the equation of trend line be y = a + b.x .... (i)

Here n = 7 (odd), one middle year [ ] and h = 1

Year	**IMR (y)**	x	x^2	$x.y$
1980	10	–3	9	–30
1985	7	–2	4	–28
1990	5	–1	1	–5

Year	IMR (y)	x	x^2	$x.y$
1995	4	0	0	0
2000	3	1	1	3
2005	1	2	4	4
2010	0	3	9	0
Total	**30**	**0**	**28**	**–56**

The normal equations are $\sum y = na + b \sum x$

As, $\sum x = 0$, $a =$ ☐

Also, $\sum xy = a \sum x + b \sum x^2$

As, $\sum x = 0$, $b =$ ☐

$\therefore$ The equation of trend line is $y =$ ☐

2) Obtain trend values for data in Problem 1 using 3 - yearly moving averages.

Solution :

Year	IMR	3 yearly moving total	3 yearly moving average (trend value)
1980	10	-	-
1985	7	☐	7.33
1990	5	16	☐
1995	4	12	4
2000	3	8	☐
2005	1	☐	1.33
2010	0	-	-

3) **Fit equation of trend line for the data given below.**

Year	**Production (y)**	x	x^2	$x.y$
2006	19	–9	81	–171
2007	20	–7	49	–104
2008	14	–5	25	–70
2009	16	–3	9	–48
2010	17	–1	1	–17
2011	16	1	1	16
2012	18	3	9	54
2013	17	5	25	85
2014	21	7	49	147
2015	19	9	81	171
Total	**177**	**0**	**330**	**33**

Let the equation of trend line be $y = a + bx$ .... (i)

Here $n =$ ☐ (even),

two middle years are ☐ and 2011, and $h =$ ☐

The normal equations are $\sum y = na + b \sum x$

As $\sum x = 0$, $a =$ ☐

Also, $\sum xy = a \sum x + b \sum x^2$

As $\sum x = 0$, $b =$ ☐

Substitute values of a and b in equation (i) the equation of trend line is ☐

To find trend value for the year 2016,

put $x =$ ☐ in the above equation.

$y =$ ☐

4) **Complete the table using 4 yearly moving average method.**

Year	Production	4 yearly moving total	4 yearly centered total	4 yearly centered moving average (trend values)
2006	19		-	-
		☐		
2007	20		-	☐
		72		
2008	17		142	17.75
		70		
2009	16		☐	17
		☐		
2010	17		133	☐
		67		
2011	16		☐	☐
		☐		
2012	18		140	17.5
		72		
2013	17		147	18.375
		75		
2014	21		-	-
		-		
2015	19		-	-

■■■

5. Index Number

Points to be remembered

- **Definition of Index Numbers :**

  An Index Number is a statistical measure of changes in a variable or a group of variables with respect to time, geographical location, or some other characteristic such as production, income, etc.

  An Index Number is used for measuring changes in some quantity that can not be measured directly.

  An Index Number is a single ratio, usually expressed as percentage, that measures aggregate (or average) change in several variables between two different times, places, or situations.

  After reading the above definitions, we can conclude that an Index Number is an 'economic indicator' of business activities.

- **Types of Index Numbers :**

  Following are three major types ot index numbers.

  1) Price Index Number :

  Price index numbers measure changes in the level of prices in the economy. It compares the price of the current year, with that of the base year to indicate the relative variation. It is a very good measure of inflation in the economy.

  2) Quantity Index Number :

  As the name suggests, quantity index numbers measure changes in the quantities of goods between the two specified years. This can be the number of goods produced, sold, consumed, etc. It is a good indication of the output of an economy.

  3) Value Index Number :

  A value index number is the ratio of the aggregate value of a given commodity (or a group of commodities) in the current year and its value in the base year. A value index number combines prices and quantities by taking the product of price and quantity as the value. The value index number thus measures the percentage change in the value of a commodity or a group of commodities during the current year in comparison to its value during the base year.

  a) Terminology :

  - **Base Period :** The base period of an index number is the period against which comparisons are made. For example, the Central Statistical Organisation (CSO) is constructing the Consumer Price Index by taking 2010 as the base year. It means that the prices in 2015 are compared with 2010 prices by taking them as 100. The base period is indicated by subscript Zero.

- **Current Period :** The present period is called the current period of an index number. An index number measures the changes between the base period and the current period. The current period is indicated by subscript 1.

  Note : The period used in index numbers can be a day, a month, or a year. We shall use a year as the period in our study.

b) Notation :

p_0 : Price of a commodity in the base year.

q_0 : Quantity (produced, purchased, or consumed) of a commodity in the base year.

p_1 : Price of a commodity in the current year.

q_1 : Quantity (produced, purchased, or consumed) of a commodity in the current year.

w : Weight assigned to a commodity according to its relative importance in the group.

I : Simple index number. It is also called the price relative. It is given by.

P_{01} : Price index for the current year with respect to the base year.

Q_{01} : Quantity index for the current year with respect to the base year.

V_{01} : Value index for the current year with respect to the base year.

- **Construction of Index Numbers :**

Index number are constructed by the following two methods.

1) Simple Aggregate Method.
2) Weighted Aggregate Method.

Let us now learn how index numbers are constructed by these two methods.

1) Simple Aggregate Method :

a) Simple Aggregate Method to find Price Index Number.

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

b) Simple Aggregate Method to find Quantity Index Number.

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100$$

c) Simple Aggregate Method to find Value Index Number.

$$V_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \times 100$$

2) Weighted Aggregate Method :

Weights are usually defined in terms of quantities in the weighted aggregate method.

$$P_{01} = \frac{\Sigma\, p_1 W}{\Sigma\, p_0 W} \times 100$$

Following are most popular price index numbers constructed by the weighted aggregate method.

a) Laspeyre's Price Index Number

$$P_{01}(\text{L}) = \frac{\Sigma\, p_1\, q_0}{\Sigma\, p_0\, q_0} \times 100$$

Note : This construction uses base year quantities as weights.

b) Paasche's Price Index Number

$$P_{01}(\text{P}) = \frac{\Sigma\, p_1\, q_1}{\Sigma\, p_0\, q_1} \times 100$$

Note : This construction uses current year quantities as weights.

c) Dorbish-Bowley's Price Index Number

$$P_{01}(\text{D} - \text{B}) = \frac{\dfrac{\Sigma\, p_1\, q_1}{\Sigma\, p_0\, q_1} + \dfrac{\Sigma\, p_1\, q_1}{\Sigma\, p_0\, q_1}}{2} \times 100$$

$$P_{01}(\text{D} - \text{B}) = \frac{p_{01}\,(\text{L}) + p_{01}\,(\text{P})}{2}$$

Note : Dorbish- Bowley's Price Index is arithmetic mean mean of Laspeyre's and Paasche's price Index Number.

d) Fisher's Ideal Price Index Number

$$P_{01}(\text{F}) = \sqrt{\frac{\Sigma\, p_1\, q_1}{\Sigma\, p_0\, q_1} \times \frac{\Sigma\, p_1\, q_1}{\Sigma\, p_0\, q_1}} \times 100$$

$$P_{01}(\text{F}) = \sqrt{p_{01}\,(\text{L}) \times p_{01}\,(\text{P})}$$

Note : Fisher's Ideal Price Index is geometric mean of Laspeyre's and Paasche's price Index Number.

e) Marshall-Edgeworth's Price Index Number

$$P_{01}(\text{M - E}) = \frac{\Sigma\, p_1\,(q_1 + q_0)}{\Sigma\, p_0\,(q_1 + q_0)} \times 100$$

$$P_{01}(\text{M - E}) = \frac{\Sigma\, p_1 q_1 + \Sigma\, p_1 q_0}{\Sigma\, p_0 q_1 + \Sigma\, p_0 q_0} \times 100$$

f) Walsh's Price Index Number

$$P_{01}(\text{P}) = \frac{\Sigma\, p_1 \sqrt{q_1 \times q_0}}{\Sigma\, p_0 \sqrt{q_1 \times q_0}} \times 100$$

- **Cost of Living Index Number :**

Cost of Living Index Number, also known as Consumer Price Index Number, is an index number of the cost of buying goods and services in day-to-day life for a specific consumer class. Different classes of consumers show different patterns of consumption of goods and services. As a result, a general index number cannot reflect changes in cost of living for a specific consumer class.

For example, cost of living index numbers for rural population are different from cost of living index numbers for urban population. The goods and services consumed by members of different consumer classes can be different and therefore cost of living index numbers calculated for different consumer classes can be based on costs of different sets of goods and services.

Methods of constructing Cost of Living Index Numbers :

1) **Aggregative Expenditure Method (Weighted Aggregate Method) :** This method uses quantities consumed in base year as weights, so that Cost of Living Index Number is defined as follows.

$$\text{CLI} = \frac{\text{Total expendutre in current year}}{\text{Total expendutre in base year}}$$

$$\therefore\ \text{CLI} = \frac{\Sigma\, p_1\, q_0}{\Sigma\, p_0\, q_0} \times 100$$

The above formula is similar to that of a weighted Index Number. Do you recognize that Index Number?

2) **Family Budget Method (Weighted Relative Method) :** Cost of Living Index Number is defined as follows.

$$\text{CLI} = \frac{\Sigma\, IW}{\Sigma\, W} \text{ where } I = \frac{p_1}{q_0} \times 100 = \text{price relative for current year}$$

And $W = p_0\, q_0$ = base year weightage.

Uses of Cost of Living Index Number :

1) Cost of Living Index Number is used to regulate the dearness allowance or the grant of bonus to employees in order to enable them bear the increased cost of living.
2) Cost of Living Index Number is used for settling dispute related to salaries and wages.
3) Cost of Living Index Number is used in calculating purchasing power of money.

$$\text{Purchasing power of money} = \frac{1}{\text{Cost of Living Index Number}}$$

4) Cost of Living Index Number is used in determining real wages.

$$\text{Purchasing power of money} = \frac{\text{Money wages}}{\text{Cost of Living Index Number}} \times 100$$

5) Cost of Living Index Numbers are widely used in negotiations of wages in wage contracts.

Solved Examples :

Ex. 1) Calculate the price index number for the following data using the Simple Aggregate Method. Take 1995 as the base year.

Commodity	**A**	**B**	**C**	**D**	**E**
Price (in Rs.) in 1995	42	30	54	70	120
Price (in Rs.) in 2005	60	55	74	110	140

Solution :

Commodities	**Price in 1995 (Base year)** p_0	**Price in 2005 (Current year)** p_1
A	42	60
B	30	55
C	54	74
D	70	110
E	120	140
Total	$\Sigma p_0 = 130$	$\Sigma p_1 = 439$

Price Index Number is then given by

$$P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \times 100$$

$$= \frac{439}{316} \times 100$$

$$= 138.92$$

Interpretation : If the price of a commodity was Rs.100 in the year 1995, then the price of the same commodity is approximately Rs.138 in the year 2005. Hence, the overall increase in the price level is 38% in ten years.

Ex. 2) Calculate the Quantity Index Number for the following data using Simple Aggregate Method .

Commodity	**I**	**II**	**III**	**IV**	**V**
Base Year Quantities	140	120	100	200	225
Current Year Quantities	100	80	70	150	185

Solution : We first tabulate the data in the following tabular form.

Commodities	**Quantity (Base year)** q_0	**Quantity (Current year)** q_1
I	140	100
II	120	80
III	100	70
IV	200	150
V	225	185
Total	$\Sigma q_0 = 785$	$\Sigma q_1 = 585$

Quantity Index Number is then given by

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100$$

$$= \frac{585}{785} \times 100$$

$$= 74.52$$

This means that the output in terms of quantity decreases by approximately 26% in current year from base year.

Ex. 3) Calculate the Value Index Number for the following data using the Simple Aggregate Method.

Commodities	Base year		Current year	
	Price Rs. p_0	Quantity (units) q_0	Price Rs. p_1	Quantity (units) q_1
P	10	6	60	7
Q	20	4	70	6
R	30	7	80	8
S	40	8	90	9
T	50	3	100	5

Solution :

Commodity	Base year		Current year			
	p_0	q_0	p_1	q_1	$p_0 q_0$	$p_1 q_1$
P	10	6	60	7	60	420
Q	20	4	70	6	80	420
R	30	7	80	8	210	640
S	40	8	90	9	320	810
T	50	3	100	5	150	500
Total					$\Sigma p_0 q_0 = 820$	$\Sigma p_1 q_1 = 2790$

Value Index Number is given by

$$V_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_1 q_0} \times 100$$

$$= \frac{2790}{820} \times 100$$

$$= 340.24$$

Ex. 4) Find x if the Price Index Number by Simple Aggregate Method is 125.

Commodity	P	Q	R	S	T
Base Year Price (in Rs.)	8	12	16	22	18
Current Year Price (in Rs.)	12	18	x	28	22

Solution :

Commodity	P	Q	R	S	T	Total
Base Year Price (in Rs.)	8	12	16	22	18	76
Current Year Price (in Rs.)	12	18	x	28	22	$80 + x$

$\Sigma p_0 = 76,\ \Sigma p_1 = 80 + x$

$$p_{01} = \frac{\Sigma p_1}{\Sigma p_1} \times 100$$

$$125 = \frac{80 + x}{76} \times 100$$

$$\frac{125 \times 76}{100} = 80 + x$$

$\therefore\ x = 15$

Ex. 5) Calculate Laspeyre's, Paasche's, Dorbish-Bowley's and Marshall-Edgeworth's Price Index Numbers for the following data.

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
A	8	20	11	15
B	7	10	12	10
C	3	30	5	25
D	2	50	4	35

Solution :

Commodity	Base year		Current year		$p_0 q_0$	$p_0 q_1$	$p_1 q_0$	$p_1 q_1$
	p_0	q_0	p_1	q_1				
A	8	20	11	15	160	120	220	165
B	7	10	12	10	70	70	120	120
C	3	30	5	25	90	75	150	125
D	2	50	4	35	100	70	200	140
Total					**420**	**335**	**690**	**550**

$\Sigma p_0 q_0 = 420,\ \Sigma p_0 q_1 = 335,\ \Sigma p_1 q_0 = 690,\ \Sigma p_1 q_1 = 550$

a) Laspeyre's Price Index Number

$$p_{01}(\text{L}) = \frac{\Sigma\, p_1 q_0}{\Sigma\, p_0 q_0} \times 100$$

$$p_{01}(\text{L}) = \frac{690}{420} \times 100 \qquad = 164.29$$

b) Paasche's Price Index Number

$$p_{01}(\text{P}) = \frac{\Sigma\, p_1 q_1}{\Sigma\, p_0 q_1} \times 100$$

$$p_{01}(\text{P}) = \frac{550}{335} \times 100 \qquad = 164.29$$

c) Dorbish-Bowley's Price Index Number

$$p_{01}(\text{D - B}) = \frac{p_1\,(\text{L}) + p_{01}\,(\text{P})}{2}$$

$$p_{01}(\text{D - B}) = \frac{164.29 + 164.18}{2} = 164.23$$

d) Marshall-Edgeworth's Price Index Number

$$p_{01}(\text{M - E}) = \frac{\Sigma\, p_1 q_1 + \Sigma\, p_1 q_0}{\Sigma\, p_0 q_1 + \Sigma\, p_1 q_0} \times 100$$

$$p_{01}(\text{M - E}) = \frac{550 + 690}{335 + 420} \times 100 = 164.24$$

Ex. 6) If $\Sigma p_0 q_0 = 420$, $\Sigma\, p_0\, q_1 = 200$, $\Sigma\, p_1\, q_0 = 350$, $\Sigma\, p_1\, q_1 = 460$ find (a) Laspeyre's, (b) Paasche's, (c) Dorbish-Bowley's and (d) Marshall-Edgeworth's Price Index Number.

Solution : a) Laspeyre's Price Index Number

$$p_{01}(\text{L}) = \frac{\Sigma\, p_1 q_0}{\Sigma\, p_0 q_0} \times 100$$

$$p_{01}(\text{L}) = \frac{350}{140} \times 100 \qquad = 250$$

b) Paasche's Price Index Number

$$p_{01}(\text{P}) = \frac{\Sigma\, p_1 q_1}{\Sigma\, p_0 q_1} \times 100$$

$$p_{01}(\text{P}) = \frac{460}{200} \times 100 = 230$$

c) Dorbish-Bowley's Price Index Number

$$p_{01}(\text{D - B}) = \frac{p_{01}\,(\text{L}) + p_{01}\,(\text{P})}{2}$$

$$p_{01}(\text{D - B}) = \frac{250 + 230}{2} = 240$$

d) Marshall-Edgeworth's Price Index Number

$$p_{01}(\text{M - E}) = \frac{\Sigma\, p_1 q_1 + \Sigma\, p_1 q_0}{\Sigma\, p_0 q_1 + \Sigma\, p_1 q_0} \times 100$$

$$p_{01}(\text{M - E}) = \frac{460 + 350}{140 + 200} \times 100 = 238.24$$

Ex. 7) Construct the Cost of Living Index Number for the following data.

Group	**Base year**		**Current year**
	Price	**Quantity**	**Price**
Food	120	15	170
Clothing	150	20	190
Fuel and Lighting	130	30	220
House Rent	160	10	180
Miscellaneous	200	12	200

Commodity	**Base year**		**Current year**	$p_0 q_0$	$p_1 q_0$
	p_0	q_0	p_1		
Food	120	20	170	1800	2550
Clothing	150	10	190	3000	3800
Fuel and Lighting	130	30	220	3900	6600
House Rent	160	10	180	1600	1800
Miscellaneous	200	12	200	2400	2400
Total				**12700**	**17150**

Solution : Cost of Living Index Number by Aggregated Expenditure Method

$$\text{CLI} = \frac{\Sigma\, p_1 q_0}{\Sigma\, p_0 q_0} \times 100$$

$$= \frac{171500}{12700} \times 100 = 135.04$$

Ex. 8) Find y if the cost of living index is 200.

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	180	120	160	300	200
W	4	5	3	y	2

Solution :

Group	I	W	IW
Food	180	4	720
Clothing	120	5	600
Fuel and Lighting	160	33	480
House Rent	300	y	$300y$
Miscellaneous	200	2	400
Total		$y + 14$	$300y + 2200$

$\Sigma\, W = y + 14,\ \Sigma\, IW = 300y + 2200$

Cost of Living Index Number by family Budget Method

$$\text{CLI} = \frac{\Sigma\, IW}{\Sigma\, W}$$

$$200 = \frac{300\, y + 2200}{y + 14}$$

$\therefore\ y = 6$

Problems for Practice

Q. 1 A) Select and write the most appropriate answer from the given alternatives for each sub-question :

1) If p_{01} (L) = 90, and p_{01} (P) = 40, then p_{01} (F) = ...

a) 60 b) 65 c) 40 d) 3600

2) Price Index Number by Weighted Aggregate Method is given by

a) $\dfrac{\Sigma\, p_1 w}{\Sigma\, p_0 w} \times 100$ b) $\dfrac{\Sigma\, p_0 w}{\Sigma\, p_1 w} \times 100$

c) $\dfrac{\Sigma\, p_1 w}{\Sigma\, p_0 w} \times 100$ d) $\dfrac{\Sigma\, p_0 w}{\Sigma\, p_1 w} \times 100$

3) Quantity Index Number by Weighted Aggregate Method is given by

a) $\frac{\Sigma q_1 w}{\Sigma q_0 w} \times 100$ b) $\frac{\Sigma q_0 w}{\Sigma q_1 w} \times 100$

c) $\frac{\Sigma q_1 w}{\Sigma q_0 w} \times 100$ d) $\frac{\Sigma q_0 w}{\Sigma q_1 w} \times 100$

4) The Cost of Living Index Number using Aggregate Expenditure Method is given by

a) $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$ b) $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$

c) $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ d) $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$

5) If Laspeyre's and Drobish-Bowley's Price Index Number are 160 and 164 respectively then Paasche's Price Index Number is ....

a) 160   b) 168   c) 164   d) 170

B) State whether the following statements are True or False.

1) Dorbish-Bowley's Price Index Number is square root of product of Laspeyre's and Paasche's Index Numbers.

2) $\frac{\Sigma p_1 \sqrt{q_1 \times q_0}}{\Sigma p_0 \sqrt{q_1 \times q_0}} \times 100$ is Marshall-Edgeworth's Price Index Number.

3) Laspeyre's Price Index Number uses current year's quantities as weights.

C) Fill in the following blanks.

1) Fisher's Price Index Number is given by ....

2) Dorbish-Bowley's Price Index is .... of Laspeyre's and Paasche's price Index Number.

3) If Laspeyre's and Paasche's Price Index Numbers are 24 and 16 respectively then Dorbish-Bowley's Price Index Number is ....

4) If $\Sigma W = 20$ then $\Sigma IW = 2580$ then Cost of Living Index Number (CIL) = ...

5) The time period for which an index number is determined is known as ....

Q. 2) **Calculate the price index number for the following data using the Simple Aggregate Method. Take 2000 as the base year.**

Commodity	**A**	**B**	**C**	**D**	**E**
Price (in Rs.) in 2000	30	35	45	55	25
Price (in Rs.) in 2003	30	50	70	75	40

Q. 3) **Calculate the Quantity Index Number for the following data using Simple Aggregate Method. take year 2000 as the base year.**

Commodity	**I**	**II**	**III**	**IV**	**V**	**VI**
Quantity in 2000	30	55	65	70	40	90
Quantity in 2004	40	60	70	90	55	95

Q. 4) **Find the value Index Number using Simple Aggregative Method.**

Commodity	**Base year**		**Current year**	
	Price	**Quantity**	**Price**	**Quantity**
I	20	42	22	45
II	35	60	40	58
III	50	22	55	24
IV	60	56	70	62
V	25	40	30	41

Q. 5) **Find x in the following table if the Aggregate Price Index Number for year 1998 with respect to Base Year 1995 is 120.**

Commodity	**I**	**II**	**III**	**IV**
Price in 1995	6	125	x	4
Price in 1998	8	18	28	6

Q. 6) **Calculate (a) Laspeyre's (b) Paasche's (c) Drobish-Bowley's and (d) Marshall – Edgworth's Price Index Number for the following data.**

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
P	12	20	18	24
Q	14	12	21	16
R	8	10	12	18
S	16	15	20	25

Q. 7) If $\Sigma p_0 q_0 = 120$, $\Sigma\ p_0\, q_1 = 160$, $\Sigma\ p_1\, q_0 = 200$, $\Sigma\ p_1\, q_1 = 140$

find (a) Laspeyre's, (b) Paasche's, (c) Dorbish-Bowley's and (d) Marshall-Edgeworth's Price Index Number.

Q. 8) If Dorbish-Bowley's and Fisher's Price Index Numbers are 5 and 4, respectively, then find Laspeyre's and Paasche's Price Index Numbers.

Q. 9) If Laspeyre's Price Index Number is four times Paasche's Price Index Number, then find the relation between Dorbish-Bowley's and Fisher's Price Index Numbers.

Q. 10) **Construct the Cost of Living Index Number for the following data.**

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	70	90	100	60	80
w	5	3	2	4	6

Q. 11) **The following table gives the base year weightage (W) and current year price relative (I) for five commodities. Calculate the Cost of Living Index Number.**

Group	Base year		Current year
	Price	Quantity	Price
Food and Clothing	40	3	70
Fuel and Lighting	30	5	60
House Rent	50	2	50
Miscellaneous	60	3	90

Q. 12) Find x if the cost of living index is 150.

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	180	120	300	100	160
W	4	5	6	x	3

Q. 13) Find x in the following table if Laspeyre's and Paasche's Price Index Numbers are equal.

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
A	2	10	2	5
B	2	5	x	2

Solution :

Commodity	Base year		Current year		$p_0 q_0$	$p_0 q_1$	$p_1 q_0$	$p_1 q_1$
	p_0	q_0	p_1	q_1				
A	2	10	2	5	20	10	20	10
B	2	5	x	2	10	4	$5x$	$2x$
Total					30	14	$\square$	$10 + 2x$

Given that p_{01} (L) = p_{01} (L)

$$\frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100 = \frac{\square}{\sum p_0 q_1} \times 100$$

$$\frac{20 + 5x}{\square} = \frac{10 + 2x}{14}$$

$$\therefore x = \square$$

Q. 14) The Price Index Number for year 2004, with respect to year 2000 as base year, is known to be 130. Find the missing numbers in the following table if sum of base year price is 320.

Commodity	**A**	**B**	**C**	**D**	**E**	**F**
Price (in Rs.) in 2000	40	50	30	x	60	100
Price (in Rs.) in 2005	50	70	30	85	y	115

Solution :

Commodities	**Price in 2000 (Base Year) p_0**	**Price in 2005 (Current Year) p_{01}**
A	40	50
B	50	70
C	30	30
D	x	85
E	60	y
F	100	115
	$280 + x$	$350 + y$

Given that $\Sigma p_0 = 320$

$280 + x = 320$

$\therefore x = \boxed{}$

Price Index Number $\Sigma p_{01} = \dfrac{\Sigma p_1}{\boxed{}} \times 100$

$130 = \dfrac{\boxed{}}{320} \times 100$

$416 = 350 + y$

$\therefore y = \boxed{}$

■■■

6. Linear Programming

Linear in equations :

A linear equation in two variables namely $ax + by + c = 0$, where a, b, c $\in$ R and (a, b) $\neq$ (0, 0), represents a straight line. A straight line makes three disjoint parts of the plane : the points lying on the straight line and two half planes on either side, which are represented by $ax + by + c < 0$ or $ax + by + c > 0$.

The set of points $\{ (x, y)|ax + by + c < 0\}$ and $\{ (x, y)\ ax + by + c > 0 \}$ are two open half planes The two sets have the common boundary $\{(x, y) \mid ax + by + c = 0\}$.

In the earlier classes, we have studied graphical solution of linear equations and linear in equations in two variables. In this chapter, we shall study these graphical solutions to find the maximum/minimum value of a linear expression

Let's Learn :

6.1 Linear Programming Problem (L.P.P.)

Linear programming is used in industries and government sectors where attempts are made to increase the profitability or efficiency and to reduce wastage. These problems are related to efficient use of limited resources like raw materials, man-power, availability of machine time, cost of material and so on.

Meaning of L.P.P.

Linear implies all the mathematical functions containing variables of index one.

A L.P.P. may be defined as the problem of maximizing or minimizing a linear function subject to linear constraints. These constraints may be equations or in equations.

Now, we formally define the terms related to L.P.P. as follows.

1) **Decision Variables :**

   The variables involved in L.P.P. are called decision variables

2) **Objective function :**

   A linear function of decision variables which is to be optimized, i.e. either maximized or minimized, is called objective function.

3) **Constraints :**

   Conditions under which the objective function is to be optimized, are called constraints. These are in the form of equations or in equations.

4) **Non-negativity constraints :**

   In some situations, the values of the variables under considerations may be positive or zero due to the imposed conditions. These constraints are referred as non-negativity constraints.

Mathematical Formulation of L.P.P.

Step 1 : Identify the decision variables as (x, y) or (x_1, x_2)

Step 2 : Identify the objective function and write it as mathematical expression in terms of decision variables.

Step 3 : Identify the different constraints and express them as mathematical equations or in equations.

The general mathematical form of L.P.P.

The L.P.P. can be put in the following form.

Maximize $z = c_1 x_1 + c_2 x_2$ ……………….. (1)

subject to the constraints.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21} x_1 + a_{22} x_2 = b_2 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 = b_m \end{array}\right\} \quad \cdots\cdots\cdots \quad (2)$$

and each $x_i \geq 0$ *f* or i = 1, 2 ....(3)

1) The linear function in (1) is called the objective function.

2) Conditions in (3) are called non-negativity constraints.

Note :

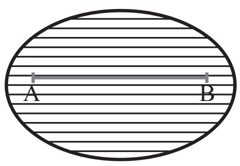
i) We shall study L.P.P. with only two variables.

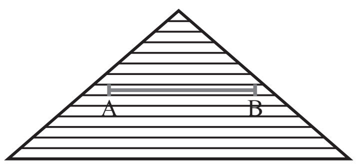
ii) We shall restrict ourselves to L.P.P. involving non-negativity constraints.

6.2.1 Convex set and feasible region.

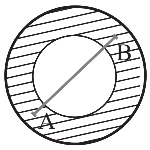
Definition : A set of points in a plane is said to be a convex set if the line segment joining any two points of the set entirely lies within the same set.

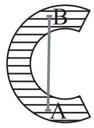
The following sets are convex sets





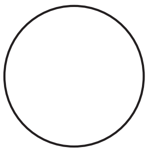
The following sets are convex sets

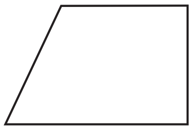




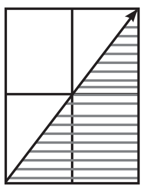
Note :

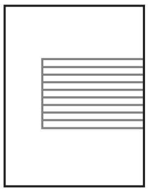
i) The convex sets may be bounded. Following are bounded convex sets.





ii) Convex sets may be unbounded Following are unbounded conves sets





Solution of LLP :

There are two methods to find the solution of L.P.P. 1) Graphical method 2) Simplex method.

Note : We shall restrict ourselves to graphical method.

Some definitions :

1) **Solution :** A set of values of variable which satisfies all the constraints of the LPP, is called the solution of the LPP.
2) **Feasible Solution :** Solution which satisfy all constraints is called feasible solution.
3) **Optimum feasible solution :** A feasible solution which optimizes i.e. either maximizes or minimizes the objective function of LPP is called optimum feasible solution.
4) **Feasible Region :** The common region determined by all the constraints and non-negativity restrictions of the linear programming problem is called the feasible region.

Note : The boundaries of the region may or may not be included in the feasible region.

Theorems (without proof) :

Theorem 1 : The set of all feasible solutions of LPP is a convex set.

Theorem 2 : The objective function of LPP attains its optimum value (either maximum or minimum) at, at least one of the vertices of convex polygon.

Note : If a LPP has optimum solutions at more than one point then the entire line joining those two points will give optimum solutions. Hence the problem will have infinite solutions Solution of LPP by Corner point method (convex polygon theorem) Algorithm :

Steps :

i) Convert all in equation of the constraints into equations.

ii) Draw the lines in xy plane, by using x intercept and y intercept of the line from its equation.

iii) Locate common region indicated by the constraints. This common region is called feasible region.

iv) Find the vertices of the feasible region.

v) Find the value of the objective function z at all vertices of the feasible region.

vi) If the objective function is of maximization (or minimization) type, then the coordinates, of the vertex (Vertices) for which z is maximum (or minimum) gives (give) the optimum solution/solutions.

Working Rule to formulate the LPP.

Step 1 : Identify the decision variables and assign the symbols x, y or $x1$, $x2$ to them. Introduce non-negativity constraints.

Step 2 : Identify the set of constraints and express them as linear inequations in terms of the decision variables.

Step 3 : Identify the objective function to be optimized (ie. maximized on minimized) and express it as a linear function of decisions variables.

*Let R be the feasible region (convex polygon) for a LPP and let $z = ax + by$ be the objective function then the optimum value (maximum or minimum) of z occurs at, at least one of the corner points (vertex) of the feasible region.

Corner point method for solving the LPP graphically.

Step 1 : Find the feasible region of the LPP.

Step 2 : Determine the vertices of the feasible region either by inspection or by solving the two equations of the lines intersecting at that points.

Step 3 : Find the value of the objective function z, at all vertices of feasible region.

Step 4 : Determine the feasible solution which optimizes the value of the objective function.

Working rule to formulate and solve the LPP Graphically.

- Identify the decision variables and assign the symbols x, y or $x1$, $x2$ to them.
- Identify the objective function (maximized or minimized) and express it as a linear function of decision variables.
- Convert in equations (constraints) into equations, find out intercept points on them.
- Draw the graph.
- Identify the feasible region (convex polygon) of the L.P.P. and shade it.
- Find all corner points of the feasible region.
- Find the value of z at all the corner points.
- State the optimum value of z (maximum or minimum).

Solved Problems :

Ex. 1) A manufacturer produces bicycles and tricycles, each of which must be processed through two machines, A and B. Machine A has maximum of 120 hours available and machine B has a maximum of 180 hours available. Manufacturing a bicycle requires 4 hours on machine A and 10 hours on machine B. Manufacturing a tricycle requires 6 hours on machine A and 3 hours on machine B. If profits are Rs. 65 for a bicycle and Rs. 45 for a tricycle, formulate L.P.P. to maximize profit.

Solution :

Let Z be the profit, which can be made by manufacturing and selling x tricycles and y bicycles. $\therefore\ x \geq 0, y \geq 0$.

$$\left.\begin{array}{l} \text{Total profit} = z = 45x + 65y \\ \text{Maximize } Z = 45x + 65y \end{array}\right\}$$

Tricycles (x)

6 hours on A — 3 hours on B

Bicycles (y)

4 hours on A — 10 hours on B

Machine	Tricyles (x)	Bicycles (y)	Availability
A	6	4	120
B	3	10	180

From the above table, remaining conditions are

$$\left.\begin{array}{l} 6x + 4y \leq 120 \\ 3x + 10y \leq 180 \end{array}\right\} \text{(Constraints)}$$

$\therefore$ The required formulated L.P.P. is as follows.

Maximize $z = 45x + 65y$ (objective function)

Subject to $6x + 4y \leq 120$

$3x + 10y \leq 180$)

$x, y \geq 0 \rightarrow$ (non negativity Constraints)

Ex. 2) Diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1500 calories. Two foods F_1 and F_2 cost Rs. 50 and Rs. 75 per unit respectively. Each unit of food F_1 contains 200 units of Vitamins, 1 unit of minerals and 40 calories, whereas each unit of food F_2 contain 100 units of vitamins, 2 units of minerals and 30 calories. Formulate the above problem as L.P.P. to satisfy sick person's requirements at minimum cost.

Solution :

Let x units of food F_1 and y units of food F_2 be fed to sick persons to meet his requirements at minimum cost $x \geq 0$, $y \geq 0$

F_1

200 units Vitamins | 1 units minerals | 40 Calories

F_2

100 units Vitamins | 2 units minerals | 30 Calories

Food / Product	F_1 (x) Per Unit	F_2 (y) Per Unit	Minimum requirement
Vitamins	200	100	4000
Mineral	1	2	50
Calories	40	30	1500
Cost/Unit Rs.	50	75	

sick person's problem is to determine x and y so as to minimize the total cost.

Total cost = z = $50x + 75y$

Minimize $z = 50x + 75y$

The remaining conditions are

$200x + 100y \geq 4000$

$x + 2y \geq 50$

$40x + 30y \geq 1500$

where x, y denote units of food F1 and F2 respectively.

$\therefore$ $x, y \geq 0$

$\therefore$ The L.P.P. is as follows.

Minimize $z = 50x + 75y$ subject to the constraints

$200x + 100y \geq 4000$,

$x + 2y \geq 50$,

$40x + 30y \geq 1500$,

$x \geq 0$, $y \geq 0$.

Ex. 3) Rakesh wants to invest at most Rs. 45000/- in savings certificates and fixed deposits. He wants to invest at least Rs. 5000/- in savings certificates and at least Rs. 15000/- in fixed deposits. The rate of interest on savings certificates is 4% p. a. and that on fixed deposits is 7% p.a. Formulate the above problem as L.P.P. to determine maximum yearly income.

Solution :

Let Rakesh invest Rs. x in savings certificate

and Rs. y in fixed deposits

$\therefore$ $x \geq 0$, $y \geq 0$

Since he has at most Rs. 45,000/- to invest, from the given conditions,

$x + y \leq 45{,}000$, $x \geq 5000$ and $y \geq 15000$

The rate of interest on savings certificate is 4% p.a. and that on fixed deposits is 7% p.a.

$\therefore$ Total annual income = $z = 0.04x + 0.07y$

$\therefore$ The L.P.P. is

Maximize $z = 0.04x + 0.07y$

subject to

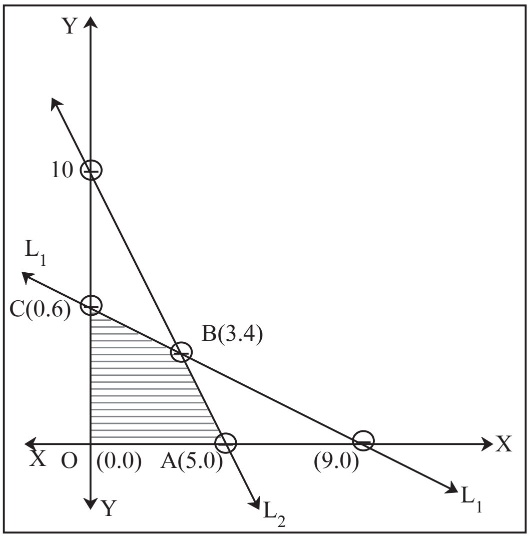
$x \geq 5000$, $y \geq 15000$, $x + y \leq 45000$, $x \geq 0$, $y \geq 0$.

Ex. 4) Maximize z = $9x + 13y$ Subject to $2x + 3y \le 18$, $2x + y \le 10$, $x \ge 0$, $y \ge 0$

Solution : To draw $2x + 3y \le 18$, and $2x + y \le 10$

Draw lines $2x + 3y = 18$, and $2x + y = 10$.

Equation of line	Intercept	Constraint type	Feasible Region
$2x + 3y - 18$	x : 9 y : 6	$\le$	Originside
$2x + 3y - 10$	x : 5 y : 10	$\le$	Originside



The common shaded region OABCO is the feasible region with vertices 0(0,0), A(5,0), B(3,4), C (0,6)

Vertex	Lines through vertex	Value of objective
A (5, 0)	$2x + y = 10$ $y = 10$	45
B (3, 4)	$2x + 3y = 18$ $2x + y = 10$	79 Maximum
C (0, 6)	$2x + 3y = 18$ $x = 10$	78
O (0, 0)	$x = 0$ $y = 0$	0

From the table, maximum value of z = 79 occurs at B (3, 4) i.e. when $x = 3$, $y = 4$.

I) Multiple choice questions :

1) The value of objective function is maximize under linear constraints

   a) at the centre of feasible region

   b) at (0, 0)

   c) at any vertex of feasible region.

   d) The vertex which is at maximum distance from (0, 0).

2) Which of the following is correct?

   a) Every LPP has an optimal solution

   b) Every LPP has unique optimal solution.

   c) If LPP has two optimal solutions then it has infinitely many optimal solutions.

   d) The set of all feasible solutions of LPP may not be a convex set.

3) Objective function of LPP is

   a) a constraint

   b) a function to be maximized or minimized

   c) a relation between the decision variables

   d) a feasible region.

4) The maximum value of $z = 5x + 3y$.

   subject to the constraints

   $3x + 5y = 15$; $5x + 2y \le 10$, x, $y \ge 0$ is.

   a) 235 b) $\frac{235}{9}$ c) $\frac{235}{19}$ d) $\frac{235}{3}$

5) The point at which the maximum value of $z = x + y$ subject to the constraints $x + 2y \le 70$, $2x + y \le 95$, $x \ge 0$, $y \ge 0$ is

   a) (36, 25) b) (20, 35) c) (35, 20) d) (40, 15)

6) Feasible region; the set of points which satisfy.

   a) The objective function.

   b) All of the given function.

   c) Some of the given constraints

   d) Only non-negative constrains

7) If the corner points of the feasible region are (0, 0), (3, 0), (2, 1) and (0, 7/3) the maximum value of $z = 4x + 5y$ is .

a) 12 b) 13 c) $\frac{35}{2}$ d) 0

8) If the corner points of the feasible region are (0, 10), (2, 2), and (4, 0) then the point of minimum $z = 3x + 2y$ is.

a) (2, 2) b) (0, 10) c) (4, 0) d) (2, 4)

9) The half plane represented by $3x + 2y \leq 0$ contains the point.

a) $(1, \frac{5}{2})$ b) (2,1) c) (0, 0) d) (5, 1)

10) The half plane represented by $4x + 3y \geq 24$ contains the point

a) (0, 0) b) (2, 2) c) (3, 4) d) (1, 1)

II) Fill in blanks :

1) Graphical solution set of the in equations $x \geq 0$, $y \geq 0$ is in ....quadrant

2) The optimal value of the objective function is attained at the ....points of feasible region.

3) The region represented by the inequality $y \leq 0$ lies in ....quadrants

4) A garage employs eight men to work in its showroom and repair shop. The constants that there must be not least 3 men in showroom and repair shop. The constrains that there must be at least 3 men in showroom and at least 2 men in repair shop are .... and .... respectively.

5) A train carries at least twice as many first-class passengers (y) as second class passengers (x). The constraint is given by..............

III) State whether each of the following is True or False :

1) The region represented by the inequalities $x \geq 0$, $y \geq 0$ lies in first quadrant.

2) The optimum value of the objective function of LPP occurs at the center of the feasible region.

3) Graphical solution set of $x \leq 0$, $y \geq 0$ in xy system lies in second quadrant.

4) The point (1, 2) is not a vertex of the feasible region bounded by

$2x + 3y \leq 6$, $5x + 3y \leq 15$, $x \geq 0$, $y \geq 0$.

5) The feasible solution of LPP belongs to only quadrant I The Feasible region of graph $x + y \leq 1$ and $2x + 2y \geq 6$ exists.

IV) Solve the following problems :

1) A Company manufactures two types of chemicals A and B. Each chemical requires two types of raw material P and Q. The table below shows number of units of P and Q required to manufacture one unit of A and one unit of B.

Chemical Raw Material	A	B	Availability
P	3	2	120
Q	2	5	160

The company gets profits of Rs. 350/- and Rs. 400/- by selling one unit of A and one unit of B respectively. Formulate the problem as LPP to maximize the profit

2) A Company manufactures two types of fertilizers F1 and F2. Each type of fertilizer requires two raw materials A and B. The number of units of A and B required to manufacture one unit of fertilizer F1 and F2 and availability of the raw materials A and B per day are given in the table below.

Fertilizer Raw Material	F_1	F_2	Availability
A	1	3	40
B	2	4	70

By selling one unit of F1 and one unit of F2, company get a profit of Rs. 500 and Rs. 750 respectively. Formulate the problem as LPP to maximize the profit.

3) The company makes concrete bricks made up of cement and sand. The weight of a concrete brick has to be at least 5 kg. cement costs Rs. 20 per kg. and sand cost Rs. 6 per kg. strength consideration dictate that a concrete brick should contain minimum 4 kg of cement and not more than 2 kg of sand. Formulate the LPP for the cost to be minimum.

4) Maximize $z = 10x + 25y$

Subject to $0 \le x \le 3$, $0 \le y \le 3$, $x + y \le 5$.

5) Minimize $z = 7x + y$ Subject to

$5x + y \ge 5$, $x + y \ge 3$, $x \ge 0$, $y \ge 0$

V) Activity based problems :

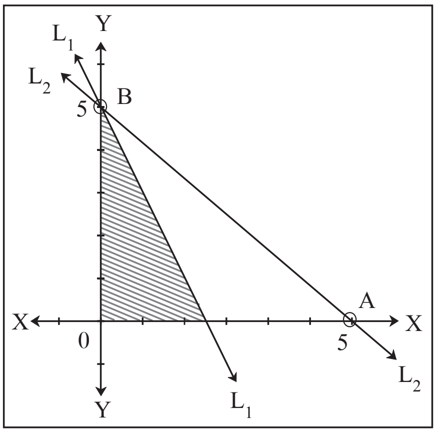
1) Find the graphical solution for the following system of linear in equations.

$8x + 5y \leq 40$, $4x + 5y \leq 40$, $x \geq 0$, $y \geq 0$

Solution to draw $8x + 5y \geq 40$

Draw line L_1 $8x + 5y = 40$

x	y	(x, y)	Sign	Region
☐	0	(☐ , 0)	≤	On Region side of line
0	☐	(0, ☐)		



To draw $4x + 5y \leq 40$ Draw line L2 : $4x + 5y = 40$

The common shaded region OABO is graphical solution, with vertices

O ☐ A ☐ B ☐

2) Shraddha wants to invest at most 25,000/- in savings certificates and fixed deposits. She wants to invest at least Rs. 10,000/- in savings certificate and at least Rs. 15,000/- in fixed deposits. The rate of interest on saving certificate is 5% per annum and that on fixed deposits is 7% per annum. Formulate the above problem as LPP to determine maximum yearly income.

Solution : Let x_1 amount (in Rs.) invest in saving certificate

x_2 : amount (in Rs.) invest in fixed deposits. $x_1 \geq 0$, $x_2 \geq 0$

From given conditions $x_1 + x_2$ [ ] 25,000

She wants to invest at least Rs. 10000/- in saving certificate

$\therefore x_1$ [ ] 10,000

Shradha want to invest at least Rs. 15,000/- in fixed deposits.

$\therefore x_2$ [ ] 15,000

Total interest = z = [ ]

Maximize z = [ ] Subject to.

[ ]

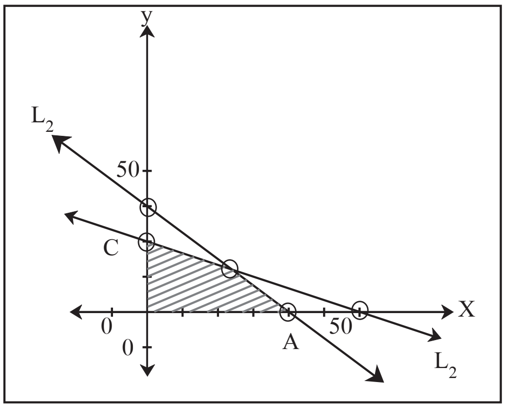
[ ]

[ ]

3) The graphical solution of LPP is shown by following figure. Find the maximum value of z = 3*x* + 2y subject to the conditions given in graphical solution.

Solution :

From Fig. 6.14. The common Shaded region OABCO is feasible region with vertices 0 [ ] , A [ ] , B(4 , 3) and C [ ]



Sr. No.	(*x*, y)	Value of $z = 3x + 2y$ at (*x*, y)
1)	O(O, O)	$z = 6$
2)	A(5, O)	$z =$ [ ]
3)	B(,)	$z =$ [ ]
4)	C(O, 3)	$z = 10$

From above table, maximum value of $z =$ ☐ occurs at point ☐ that is when $x =$ ☐ , $y =$ ☐

4) Formulate and solve the following LPP. A company manufactures bicycles and tricycles, each of which must be processed through two machines A and B. Machine A has maximum of 120 hours available and machine B has a maximum of 180 hours available. Manufacturing a Bicycle require 6 hours on machine A and 3 hours on machine B. Manufacturing a tricycle requires 4 hours on machine A and 10 hours on machine B. If profits are Rs. 180/- for a bicycle and Rs. 220/- for a tricycle, determine the number of bicycles and tricycles that should be manufactured in order to maximize the profit.

Solution :

Let x no of bicycles and y no. of tricycles be manufactured $x \geq 0$, $y \geq 0$ ....1 Total profit = z = Maximize z = The remaining conditions are............

$\therefore$ LPP is maximize z = ...

subject to $x \geq 0$, $y \geq 0$, ...

To draw $6x + 4y \leq 120$

Draw line $L_1 : 6x + 4y = 120$

x	y	(x, y)	Sign	Region
☐	0	(☐ , 0)	$\leq$	Origin Side of Line L_1
0	☐	(0 , ☐)		

To draw $3x + 10y \leq 180$

Draw line $L_2 : 3x + 10y = 180$

x	y	(x, y)	Sign	Region
☐	0	(☐ , 0)	$\leq$	☐
0	☐	(0 , ☐)		

The common shaded region 333333 is feasible region with vertices.

0(0 , 0), A ______ , B(10 , 15), C(0 , 18).

Sr. No.	(*x*, y)	Value of $z = 180x + 220y$ at (*x*, y)
1)	0(0, 0)	$z = 0$
2)	A(, o)	$z =$ ______
3)	B(,)	$z =$ ______
4)	C(0, 18)	$z =$ ______

Maximum value of z = ______ occurs at point ………..... that is when $x =$ ……..., $y =$ ………

Thus company gets maximum profit

$z =$ Rs. ______

when $x =$ ______ no of bicycles and

$y =$ ______ no of tricycles are manufactured.

■■■

7. Assignment Problem and Sequencing

7.1 Definition of Assignment Problem :

Assignment problem is a special type of problem which deals with allocation of various resources to various activities on one to one basis. It is done in such a way that the total cost or **time involved** in the process is minimum or the total profit is maximum.

Conditions :

i) Number of jobs is equal to number of machines or workers.

ii) Each worker or machine is assigned to only one job.

iii) Each worker or machine is independently capable of handling any job.

iv) Objective of the assignment is clearly specified (minimizing cost or maximizing profit)

Assignment Model :

Given n workers and n jobs with the cost of every worker for every job, the problem is to assign each worker to one and only one job so as to optimize the total cost.

Let C_{ij} be the cost of assigning i^{th} worker to j^{th} job, x_{ij} be the assignment of ith worker to j^{th} job and x_{ij} = 1, if i^{th} worker is assigned to j^{th}

job = 0, otherwise

Following table represents the cost of assigning n workers to n jobs.

Worker	Jobs					
	1	2	3	....	....	n
1	C_{11}	C_{12}	C_{13}	....	....	C_{1n}
2	C_{11}	C_{12}	C_{13}	....	....	C_{1n}
3	C_{11}	C_{12}	C_{13}	....	....	C_{1n}
- - -	- - -	- - -	- - -	- - -	- - -	- - -
n	C_{n1}	C_{n2}	....	....	....	C_{nn}

The objective is to make assignments that minimize the total cost.

Thus, an assignment problem can be represented by n × n matrix which covers all the n! possible ways of making assignments.

Assignment Problem is a special case of Linear Programming Problem.

Assignment problem can be expressed symbolically as follows :

Minimize $Z = \sum_{i=1}^{n} \sum_{i=1}^{n} C_{ij} x_{ij}$

Subject to constraints $\sum_{j=1}^{n} x_{ij} = 1,\ i = 1, 2, 3,,,,n$

(exactly one job is assigned to i^{th} worker)

$\sum_{i=1}^{n} x_{ij} = 1\ ,\ i = 1, 2, 3,,,,n$

(exactly one worker is assigned to jth job where xijtakes a value 0 or 1.

7.2 Hungarian Method :

Hungarian method is based on the following properties :

1) If a constant (positive or negative) is added to every element of any row or column in the given cost matrix, an assignment that minimizes the total cost in the original matrix also minimizes the total cost in the revised matrix.
2) In an assignment problem, a solution having zero total cost of assignment is an optimal solution. The Hungarian algorithm can be explained with the help of the following example.

Consider an example where 4 jobs need to be performed by 4 workers, one job per worker. The matrix below shows the cost of assigning a certain worker to a certain job. The objective is to minimize the total cost of assignment.

Worker	**Jobs**			
	J_1	J_2	J_3	J_4
W_1	62	63	50	72
W_2	57	35	49	60
W_3	21	49	15	56
W_4	18	19	78	23

Let us solve this problem by Hungarian method.

Step 1 : Subtract the smallest element of each row from every element of that row.

Worker	**Jobs**			
	J_1	J_2	J_3	J_4
W_1	12	13	0	22
W_2	22	0	14	25
W_3	6	34	0	41
W_4	0	1	60	5

Step 2 : Subtract the smallest element of each column from every element of that column.

Worker	Jobs			
	J_1	J_2	J_3	J_4
W_1	12	13	0	17
W_2	22	0	14	20
W_3	6	34	0	36
W_4	0	1	60	0

Step 3 : Assign through zeros.

Worker	Jobs				
			√		
W_1	12	13	[0]	17	√
W_2	22	[0]	14	20	
W_3	6	34	~~0~~	36	√
W_4	[0]	1	60	~~0~~	

Observe that third row does not contain an assignment.

Step 4 :

1) Mark (√) the row (R3).
2) Mark (√) the columns (C3) having zeros in the marked rows.
3) Mark (√) the row (R1) which contains assignment in marked column.
4) Draw straight lines through marked columns and unmarked rows.

Worker	Jobs				
			√		
W_1	12	13	[0]	17	√
W_2	22	[0]	14	20	
W_3	6	34	~~0~~	36	√
W_4	[0]	1	60	~~0~~	

All zeros can be covered using 3 lines. Therefore, number of lines required = 3 and order of matrix = 4 Hence, the number of lines required ≠ order of matrix. Therefore we continue with the next step to create additional zeros.

Step 4 :

(i) Find the smallest uncovered element (6)

(ii) Subtract this number from all uncovered elements and add it to all elements which lie at the intersection of two lines and other elements on the lines remain unchanged.

Worker	Jobs			
	J_1	J_2	J_3	J_4
W_1	6	7	0	11
W_2	22	0	20	20
W_3	6	28	0	30
W_4	0	1	66	0

Step 5 : Assigning through zeros we get.

Worker	Jobs			
	J_1	J_2	J_3	J_4
W_1	6	7	[0]	11
W_2	22	[0]	20	20
W_3	[0]	28	0	30
W_4	0	1	66	[0]

Now, each row and each column contains an assignment.

Hence, optimal solution is obtained and the optimal assignment is as follows.

Worker 1 should perform job 3, worker 2 job 2, worker 3 job 1 worker 4 job 4

i.e. W1→J3, W2→J2, W3→J1, W4→J4

Total Minimum Cost = 50 + 35 + 21 + 23 = Rs. 129

Steps of the Hungarian Method :

Following steps describe the Hungarian Method.

Step 1 : Subtract the minimum cost in each row of the cost matrix from all the elements in the respective row.

Step 2 : Subtract the minimum cost in each column of the cost matrix from all the elements in the respective column.

Step 3 : Starting with the first row, examine the rows one by one until a row containing exactly single zero is found. Make an assignment by marking ( ) that zero. Then cross (×) all other zeros in the column in which the assignment was made. This eliminates the possibility of making further assignments in that column.

Step 4 : After examining all the rows, repeat the same procedure for columns. i.e. examine the columns one by one until a column containing exactly one zero is found. Make an assignment by marking ( ) that zero. Then cross (×) all other zeros in the row in which the assignment was made.

Step 5 : Continue these successive operations on rows and columns until all the zeros have been either assigned or crossed out and there is exactly one assignment in each row and in each column. In such case optimal solution is obtained.

Step 6 : There may be some rows (or columns) without assignments i.e. the total number of marked zeros is less than the order of the cost matrix. In such case, proceed to step 7.

Step 7 : Draw the least possible number of horizontal and vertical lines to cover all zeros. This can be done as follows: i) Mark (☐) the rows in which no assignment has been made. ii) Mark (☐) the column having zeros in the marked rows. iii) Mark (☐) rows which contain assignments in marked columns. iv) Repeat 2 and 3 until the chain of marking is completed. v) Draw straight lines through marked columns. vi) Draw straight lines through unmarked rows. By this way we draw the minimum number of horizontal and vertical lines required to cover all zeros. If the number of lines is less than the order of matrix, then there is no solution. And if the minimum number of lines is equal to the order of matrix, then there is a solution and it is optimal.

Step 8 : If minimum number of lines < order of matrix, then

a) Select the smallest element not covered by any of the lines of the table.

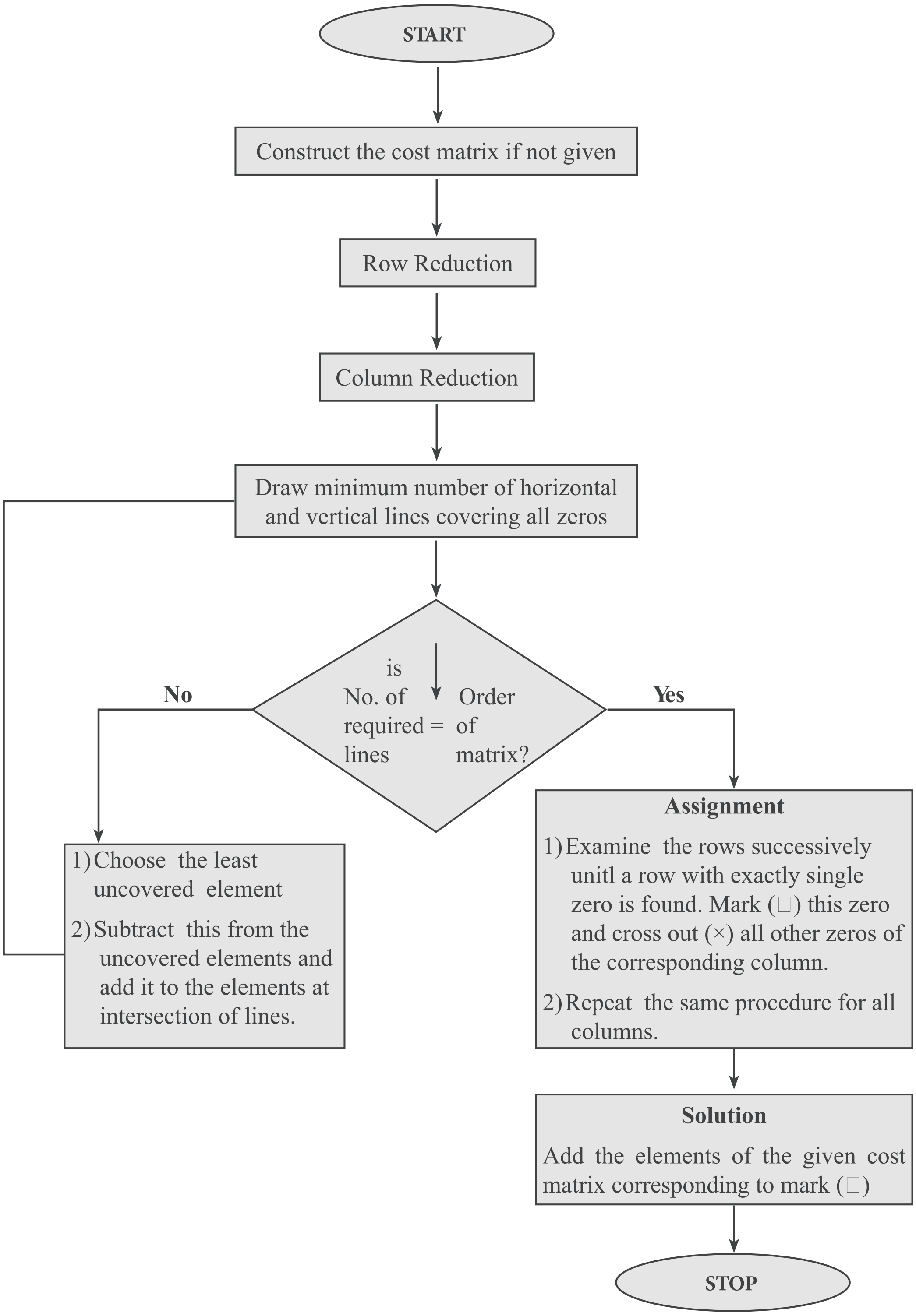
b) Subtract this value from all the uncovered elements in the matrix and add it to all those elements which lie at the intersection of horizontal and vertical lines.

Step 9 : Repeat steps 4, 5 and 6 until we get the number of lines covering all zeros equal to the order of matrix. In this case, optimal solution can be obtained.

Step 10 We now have exactly one marked ( ) zero in each row and each column of the cost matrix. The assignment schedule corresponding to these zeros is the optimal assignment.

Let's Note : The Hungarian Method was developed and published in 1955 by Harold Kuhn, who gave the name 'Hungarian Method' as the algorithm was largely based on the earlier works of two Hungarian mathematicians : Dènes KÓnig and JenÒ Egervàry.

Flow Chart of Hungarian Method :



7.3 Special Cases of Assignment Problem :

The assignment problem is generally defined as a problem of minimization. In practice, some situations are like Assignment Problem but with some variations. The following four variations are more common and can be solved using the Hungarian method.

i) **Unbalanced assignment problem :** An unbalanced assignment problem is one in which the number of resources is not equal to the number of activities i.e. the cost matrix of an assignment problem is not a square matrix (no. of rows ≠ no. of columns). An unbalanced assignment problems can be balanced by adding dummy resources/tasks (row/column) with zero costs.

ii) **Maximization Problem :** Sometimes the assignment problem may deal with maximization of the objective function. To solve such a problem, we need to convert it to minimization so that we can solve it using Hungarian Method. This conversion to minimization problem can be done in either of the following ways: (i) by subtracting all the elements from the largest element of the matrix 115 (ii) by multiplying all the elements of the matrix by '-1' Then this equivalent minimization problem can be solved using Hungarian method.

iii) **Restricted assignment problem :** An assignment problem involving restrictions on allocation due to personal, technical, legal or other reasons is called a restricted assignment problem. A restricted assignment problem does not allow some worker(s) to be assigned to some job(s). It can be solved by assigning a very high cost (or infinite cost) to the restricted cells where assignment cannot be made.

iv) **Alternative optimal solutions :** An alternate (multiple) solution exists for an assignment problem when the final assignment matrix contains more than the required number of zeros. In this case, assignments can be made through zeros arbitrarily, keeping in mind that each row and each column can contain only one assignment.

Sequencing Problem :

Introduction to Sequencing Problem :

Suppose we have two machines - A : Cutting and B : Sewing machine Suppose there are two items I and II to be processed on these machines in the order A-B. The machines can handle only one job at a time and the time taken in hours by the machines to complete the jobs is given by following table.

Machine	Item	
	I	II
A	6	3
B	3	6

Then there are two ways of completing this task

i) Processing in the order I - II

ii) Processing in the order II - I

Case I : Let us start with item I at 0 hours.

Then we get

Item	**Machine**			
	A		**B**	
	In	Out	In	Out
1	0	6	6	9

Processing of item I starts at 0 hrs and is completed at 6 hrs,

Note that during this time, though machine B is idle, it can not process job II, since cutting is required completed before sewing. Once the processing of item I is completed on machine A at 6 hours, it is shifted to machine B for sewing immediately. Machine B being idle, item I is immediately taken for processing on machine B without any wastage of time. Therefore, 'time in' for item I on machine B is 6 and time out is 6 + 3 = 9; where 3 hrs is the time required for the processing of item I on machine B. While item I is being processed on machine B. Machine A is free and hence, it can take item II for processing. Thus, item II enters machine A at 6 hrs and since it needs 3 hrs for cutting (refer to table 1) it gets out at 9 hrs from machine A. At 9 hrs machine B is available, and hence can take, item II for processing at that time. Item II requires 6 hrs on machine B, and will be out from machine B at 9 + 6 = 15 hrs. as shown below :

Item	**Machine**			
	A		**B**	
	In	Out	In	Out
I	0	6	6	9
II	6	9	9	15

Thus, the processing of items I and II in the order I - II takes 15 hours.

Case II :

Let us now see what happens if we change the order of processing the two items i.e. processing item II first and item I second, (II - I)

Repeating the same process as in case (i) we get

Item	Machine			
	A		B	
	In	Out	In	Out
I	0	6	6	9
II	6	9	9	15

Therefore, processing of items in the order II-I takes 12 hours.

Observe that :

Order of processing the items	Time required to complete the task
	In
I - II	15 hrs.
II - I	12 hrs.

By a mere change in the order of processing of the two jobs, we could save 3 hrs.

Thus, it is very important to decide the order in which the jobs should be lined up for processing so as to complete the entire schedule in minimum time.

Such type of problem where, one has to determine the order or sequence in which the jobs are to be processed through machines so as to minimize the total processing time is called a 'sequencing problem.'

Conditions :

1) No machine can process more than one job at a time.
2) Each job, once started, must be processed till its completion.
3) The processing times are independent of the order of processing the jobs.
4) Each machine is of different type.
5) The time required to transfer a job from one machine to another is negligible.

Terminology :

1) Total Elapsed Time : It is the time required to complete all the jobs i.e. the entire task. Thus, total elapsed time is the time between the beginning of the first job on the first machine till the completion of the last job on the last machine.

2) Idle Time : Idle time is the time when a machine is available but not being used, i.e. the machine is available but is waiting for a job to be processed.

General Sequencing Problem :

Let there be `n' jobs, to be performed one at a time, on each of 'm' different machines, where the order of processing on machines and the processing time of jobs on machines is known to us. Then our aim is to find the optional sequence of processing jobs that minimizes the total processing time or cost. Hence our job is to find that sequence out of (n!)m sequences, which minimizes the total elapsed time.

Notations :

A_i, B_i	:	Processing time required by i^{th} job on machine A and machine B (i =1, 2, 3...n)
T	:	Total elapsed time
X_A, X_B	:	Idle times on machines A, B from end of $(i - 1)^{th}$ job to the start of i^{th} job

Type of Sequencing problems :

I) Sequencing n jobs on two machine.

II) Sequencing n jobs on three machine.

7.4.1 Sequencing n jobs on Two Machine :

Let there be 'n' job each of which is to be processed through two machines say A and B in the order AB. Let the processing time A_1, A_2, A_3, ...A_n, B_1, B_2, B_3...B_n be given.

Algorithm to find Optimal Sequence :

1) Find out Min {Ai, Bi}
2) (a) If the minimum processing time is Ar, then process r^{th} job first.
   (b) If the minimum processing time is Bs, then process s^{th} job in the last.
3) Case of tie : Tie can be broken arbitrarily.
4) Cross off the jobs already placed in the sequence and repeat steps 1 to 3 till all the jobs are placed in the sequence.
5) Once the sequence is decided, prepare the work table and find total elapsed time.

7.4.2 Sequencing 'n' Jobs on Three Machines :

Let there be 'n' jobs each of which is to be processed through three machines say A, B and C in the order ABC. To solve this problem –

(i) first reduce it to the 'n job 2 machine' problem and determine the optimal sequence

(ii) once the sequence is determined, go back to the original 3 machines and prepare the work table for 3 machines. Conditions for reducing a 3 machine problem to a 2 machine problem :

To convert a 3 machine problem into a 2 machine problem, at least one of the following conditions must hold true.

1) The minimum processing time for machine A is greater than or equal to the maximum processing time for machine B. i.e. Min Ai$\geq$ Max Bi, i = 1, 2, 3, ... n OR

2) The minimum processing time for machine C is greater than or equal to the maximum processing time for machine B. i.e. Min Ci, $\geq$ Max Bi, i = 1, 2, 3, ... n

Let's Remember :

- Assignment Problem is a special case of LPP in which every worker or machine is assigned only one job.
- Objective of the assignment is clearly specified (minimizing cost or maximizing profit).
- Hungarian Method is used to solve a minimization assignment problem.
- **Special Cases of Assignment Problem :**
  1) **Unbalanced assignment problem :** (No. of rows $\neq$ No of columns) An unbalanced assignment problem can be balanced by adding dummy row/column with zero costs.
  2) **Maximization Problem :** Such problem is converted to minimization by subtracting all the elements from the largest element of the matrix. Then this can be solved by Hungarian method.
  3) **Restricted assignment problem :** It can be solved by assigning a very high cost (infinite cost) to the restricted cell.
  4) **Alternative optimal solutions :** If the final assignment matrix contains more than the required number of zeros, assign through zeros arbitrarily.
- **Sequencing problem :** In sequencing problems, one has to determine the order or sequence in which the jobs are to be processed through machines so as to minimize the total processing time.
- **Total Elapsed Time :** It is the time required to complete all the jobs i.e. the entire task.
- **Idle Time :** Idle time is the time when a machine is available, but is not being used. • Types of sequencing problems :
- Sequencing n jobs on Two machines :
- Sequencing n jobs on Three machines :

To convert a 3 machine problem into a 2 machine problem, at least one of the following conditions must hold true.

1) Min Ai $\geq$ Max Bi OR 2) Min Ci $\geq$ Max Bii = 1, 2, 3, ... n

If either one of the above conditions hold, introduce two fictitious machines say

• G and H such that : Gi = Ai + Bi and

Hi = Bi + Ci ,

i = 1, 2, 3,...n

If not, the problem cannot be solved.

Solved Problems :

Ex.1) A departmental store has four workers to pack their goods. The times (in minutes) required for each worker to complete the packings per item sold is given below. How should the manager of the store assign the jobs to the workers, so as to minimize the total time of packing.

Workers	**Packing of**			
	Books	Toys	Crockery	Cutlery
A	0	11	10	8
B	13	2	12	2
C	3	4	6	1
D	4	15	4	9

Solution : Let us solve this problem by Hungarian method.

Step 1 : Subtract the smallest element of each row from every element of that row.

Workers	**Packing of**			
	Books	Toys	Crockery	Cutlery
A	0	8	7	5
B	11	0	10	~~0~~
C	2	3	5	0
D	~~0~~	11	0	5

Step 2 : Since all column minimums are zero, no need to subtract anything from Columns.

Step 3 : Assigning through zeros we get,

Workers	**Packing of**			
	Books	Toys	Crockery	Cutlery
A	[0]	8	7	5
B	11	[0]	10	~~0~~
C	2	3	5	[0]
D	~~0~~	11	[0]	5

$\therefore$ Optimal assignment schedule is :

A $\rightarrow$ Books, B $\rightarrow$ Toys, C $\rightarrow$ Cutlery, D $\rightarrow$ Crockery.

Total Minimum Time $= 3 + 2 + 4 + 1 = 10$ minutes.

Ex. 2) [Unbalanced assignment problem]

A departmental head has four subordinates, and three tasks to be performed. The subordinates differ in efficiency. Estimated time for that task would take to perform each given in the matrix below How should the tasks be allotted so as to minimize the total man hours?

Job	**Man**			
	M_1	M_2	M_3	M_4
A	7	2	6	3
B	3	7	5	4
C	5	4	3	7

Solution :

Step 1 : Observe that the number of rows is not equal to number of columns in the above matrix. Therefore it is an unbalanced assignment problem. It can be balanced by introducing a dummy job D with zero cost.

Job	**Man**			
	M_1	M_2	M_3	M_4
A	7	2	6	3
B	3	7	5	4
C	5	4	3	7
D	0	0	0	0

Step 2 : Subtract the smallest element of each row from every elements of that row.

Job	**Man**			
	M_1	M_2	M_3	M_4
A	5	0	4	1
B	0	4	5	1
C	2	1	0	4
D	0	0	0	0

Step 3 : Since all the column minimums are zeros, no need to subtract anything from columns.

Step 4 : Assigning through zeros we get,

Job	Man			
	M_1	M_2	M_3	M_4
A	5	$\boxed{0}$	4	1
B	$\boxed{0}$	4	2	1
C	2	1	$\boxed{0}$	4
D	$\not{0}$	$\not{0}$	$\not{0}$	$\boxed{0}$

Optimal Solution :

Job	Man	Man hours
A	M_2	2
B	M_1	3
C	M_3	3
D	**Total**	8

Ex. 3) (Maximization Case and Alternative Optimal Solutions)

A marketing manager has list of salesmen and towns. Considering the capabilities of the salesmen and the nature of towns, the marketing manager estimates amounts of sales per month (in thousand rupees) for each salesman in each town. Suppose these amounts are as follows :

Salesman	Town				
	T_1	T_2	T_3	T_4	T_5
S_1	37	43	45	33	45
S_2	45	29	33	26	41
S_3	46	32	38	35	42
S_4	27	43	46	41	41
S_5	34	38	45	40	44

Find the assignment of salesmen to towns that will result in maximum sale.

Solution : The above maximization problem can be converted into the equivalent minimization problem by subtracting all the matrix elements from the largest element which is 46. Then the resulting matrix is.

Salesman	Town				
	T_1	T_2	T_3	T_4	T_5
S_1	9	3	1	13	1
S_2	1	17	13	20	5
S_3	0	14	8	11	4
S_4	19	3	0	5	5
S_5	12	8	1	6	2

Now, we can solve this problem by Hungarian method.

Step 1 : Subtract the smallest element of each row from every element of that row.

Salesman	Town				
	T_1	T_2	T_3	T_4	T_5
S_1	8	2	0	12	0
S_2	0	16	12	19	4
S_3	0	14	8	11	4
S_4	19	3	0	5	5
S_5	11	7	0	5	1

Step 2 : Subtract the smallest element of each column from every element of that column.

Salesman	Town				
	T_1	T_2	T_3	T_4	T_5
S_1	8	0	0	7	0
S_2	0	14	12	14	4
S_3	0	12	8	6	4
S_4	19	1	0	0	5
S_5	11	5	0	0	1

Step 3 : Draw minimum number of lines (horizontal and vertical) that are required to cover all zeros in the matrix.

	T_1	T_2	T_3	T_4	T_5
S_1	8	0	0	7	0
S_2	0	14	12	14	4
S_3	0	12	8	6	4
S_4	19	1	0	0	5
S_5	11	5	0	0	1

Therefore, number of lines required (4) < order of matrix (5) Therefore we continue with next step to create additional zeros.

Step 4 :

(i) Find the smallest uncovered elements (4).

(ii) Subtract this number from all uncovered elements and add it to all elements which lie at the intersection of two lines.

	T_1	T_2	T_3	T_4	T_5
S_1	12	0	0	7	0
S_2	0	10	8	10	0
S_3	0	8	4	2	0
S_4	23	1	0	0	5
S_5	15	5	0	0	1

Step 5 : We return to step 3 i.e. again we determine the minimum number of lines required to cover all zeros in the matrix.

	T_1	T_2	T_3	T_4	T_5
S_1	12	0	0	7	0
S_2	0	10	8	10	0
S_3	0	8	4	2	0
S_4	23	1	0	0	5
S_5	15	5	0	0	1

Number of lines required (5) = Order of matrix Therefore optimal assignment can be made.

Optimal assignment :

Optimal assignment can be made through zeros.

Note that after assigning S1→T1, each row and column more than one zeros. Therefore alternate optimal solutions exist. Assigning through zeros in different ways, we get two different assignments :

(i)

	T_1	T_2	T_3	T_4	T_5
S_1	12	[0]	~~0~~	7	~~0~~
S_2	~~0~~	10	8	10	[0]
S_3	[0]	8	4	2	~~0~~
S_4	23	1	0	[0]	5
S_5	15	5	[0]	~~0~~	1

S1→T2, S2→T5, S3→T1, S4→T4, S5→T3

Maximum Sale = 43 + 41 + 46 + 41 + 45

= 216 thousand rupees.

(ii)

	T_1	T_2	T_3	T_4	T_5
S_1	12	[0]	~~0~~	7	~~0~~
S_2	[0]	10	8	10	~~0~~
S_3	~~0~~	8	4	2	[0]
S_4	23	1	[0]	~~0~~	5
S_5	15	5	~~0~~	[0]	1

S1→T_2, S_2→T_1, S_3→T_5, S_4→T_3, S_5→T_4

Maximum Sale = 43 + 45 + 42 + 46 + 40

= 216 thousand rupees.

Observe that the amount of Maximum Sale is same in both the cases.

Ex. 4) (Restricted Assignment Problem)

Three new machines M_1, M_2, M_3 are to be installed in a machine shop. There are four vacant places A, B, C, D. Due to limited space, machine M_2 can not be placed at B.

The cost matrix (in hundred rupees) is as follows :

Machine	**Places**			
	A	B	C	D
M_1	13	10	12	11
M_2	15	-	13	20
M_3	5	7	10	6

Determine the optimum assignment schedule.

Solution :

Step 1 :

(a) Observe that the number of rows is not equal to number of columns in the above matrix. Therefore it is an unbalanced assigned problem. It can be balanced by introducing a dummy job D with zero cost. (b) Also, it is a restricted assignment problem. So we assign a very high cost '∞' to the prohibited cell.

Machine	**Places**			
	A	B	C	D
M_1	13	10	12	11
M_2	15	∞	13	20
M_3	5	7	10	6
M_4	0	0	0	0

Step 1 : Subtract the smallest element of each row from every element of that row.

Machine	**Places**			
	A	B	C	D
M_1	3	0	2	1
M_2	2	∞	0	7
M_3	0	2	5	1
M_4	0	0	0	0

Step 3 : Since all the column minimums are zeros, no need to subtract anything from columns.

Step 4 : Assigning through zeros we get,

Machine	**Places**			
	A	B	C	D
M_1	3	0	2	1
M_2	2	∞	0	7
M_3	0	2	5	1
M_4	~~0~~	~~0~~	~~0~~	0

Optimal Solution :

Machine	**Places**	**Man hours**
M_1	B	10
M_2	C	13
M_3	A	5
	Total	28

Therefore, Total Minimum Cost = 28 hundred rupees.

Ex. 5) Determine the optimal sequence of job that minimizes the total elapsed time for the date given below (processing time on machines is given in hours). Also find total elapsed time T and the idle time for three machines.

Job	**I**	**II**	**III**	**IV**	**V**	**VI**	**VII**
Machine A	3	8	7	4	9	8	7
Machine B	4	3	2	5	1	4	3
Machine C	6	7	5	11	5	6	12

Solution : Here, minA = 3, min C = 5, and max B = 5 Since min C $\geq$ max B is satisfied, the problem can be converted into a two machine problem. Let G and H be two fictions machines such that G = A + B and H = B + C Then the problem can be written as

Job	**I**	**II**	**III**	**IV**	**V**	**VI**	**VII**
Machine G	7	11	9	9	10	12	10
Machine H	10	10	7	16	6	10	15

Using the optimal sequence algorithm, the following optimal sequence can be obtained.

I	IV	VII	VI	II	III	V

Total elapsed time

Job	Machine A		Machine B		Machine C	
	In	Out	In	Out	In	Out
I	0	3	3	7	7	13
IV	3	7	7	12	13	24
VII	7	14	14	17	24	36
VI	14	22	22	26	36	42
II	22	30	30	33	42	49
III	30	37	37	39	49	54
V	37	46	46	47	54	59

$\therefore$ Total elapsed time = 59 hrs.

Idle time for machine A = 59 - 46 = 13 hrs.

Idle time for machine B = 59 - 22 = 37 hrs.

Idle time for machine C = 59 - 52 = 7 hrs.

I) Multiple choice questions :

1) In sequencing, an optimal path is one that minimizes ....

   a) Elapsed time b) Idle time

   c) Both a) and b) d) Ready time

2) The objective of sequencing problem is

   a) to find the order in which jobs are to be made

   b) to find the time required for the completing all the job on hand

   c) to find the sequence in which jobs on hand are to be processed to minimize the total time required for processing the jobs

   d) to maximize the cost

3) The Assignment Problem is solved by

a) Simplex method, b) Hungarian method

c) Vector method, d) Graphical method,

4) In solving 2 machine and n jobs sequencing problem, the following assumption is wrong.

a) No passing is allowed

b) Processing times are known

c) Handling time is negligible

d) The time of passing depends on the order of Machining

5) To use the Hungarian method, a profit maximization assignment problem requires.

a) Converting all profits to opportunity losses

b) A dummy person or job

c) Matrix expansion

d) Finding the maximum number of lines to cover all the zeros in the reduced matrix

6) The assignment problem is said to be balanced if

a) Number of rows is greater than number of columns

b) Number of rows is lesser than number of columns

c) Number of rows is equal to number of columns

d) If the entry of row is zero

7) In an assignment problem if number of rows is greater than number of columns then

a) Dummy column is added b) Dummy row is added

c) Row with cost 1 is added d) Column with cost 1 is added

8) In a 3 machine and 5 jobs problem, the least of processing times on machine A, B and C are 5, 1. and 3 hours and the highest processing times are 9, 5, and 7 respectively, then it can be converted to a 2 machine problem if order of the machines is :

a) B-A-C, b) A-B-C c) C - B – A d) Any order

9) The objective of an assignment problem is to assign

a) Number of jobs to equal number of persons at maximum cost to assign

b) Number of jobs to equal number of persons at minimum cost

c) Only the maximize cost

d) Only to minimize cost

II) Fill in blanks :

1) An assignment problem is said to be unbalanced when ....

2) For solving an assignment problem the matrix should be a ....matrix.

3) A dummy row(s) or column(s) with the cost elements as .... the matrix of an unbalanced assignment problem as a square matrix.

4) The time interval between starting the first job and completing the last. job including the idle time (if any) in a particular order by the given set of machines is called ....

5) Maximization assignment problem is transformed to minimization problem by subtracting each entry in the table from the.................. value in the table.

6) The time required for printing of four books A, B, C and D is 5, 8, 10 and 7 hours.While its data entry requires 7, 4, 3 and 6 hrs respectively. The sequence that minimizes total elapsed time is..........................

III) State whether each of the following is True or False :

1) One machine - one job is not an assumption in solving sequencing problems.

2) To convert the assignment problem into a maximization problem, the smallest element in the matrix is deducted from all other elements.

3) The Hungarian method operates on the principle of matrix reduction, whereby the cost table is reduced to a set of opportunity costs.

4) Using the Hungarian method, the optimal solution to an assignment problem is found when the minimum number of lines required to cover the zero cells in the reduced matrix equals the no of persons.

5) One of the assumptions made while sequencing n jobs on 2 machines is : two jobs must be loaded at a time on any machine.

IV) Solve the following problems :

1) Five wagons are available at stations 1, 2, 3, 4 and 5. These are required at 5 stations I, II, III, and IV and V. The mileage between various stations are given in the table below. How should the wagons be transported so as to minimize the mileage covered?

	I	II	III	IV	V
1	10	5	9	18	11
2	13	9	6	12	14
3	3	2	4	4	5
4	18	9	12	17	15
5	11	6	14	19	10

2) In the modification of a plant layout of a factory four new machines M1, M2, M3 and M4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M2 can not be placed at C and M3 can not be placed at A the cost of locating a machine at a place (in hundred rupees) is as follows.

Machines	**Location**				
	A	B	C	D	E
M1	9	11	15	10	11
M2	12	9	-	10	9
M3	-	11	14	11	7
M4	14	8	12	7	8

Find the optimal assignment schedule.

3) A machine operator has to perform two operations, turning and threading on 6 different jobs. The time required to perform these operations (in minutes) for each job is known. Determine the order in which the jobs should be processed in order to minimize the total time required to complete all the jobs. Also find the total processing time and idle times for turning and threading operations.

Job	1	2	3	4	5	6
Time for turning	3	12	5	2	9	11
Time for threading	8	10	9	6	3	1

4) An insurance company receives three types of policy application bundles daily from its head office for data entry and tiling. The time (in minutes) required for each type for these two operations is given in the following table :

Policy	1	2	3
Data Entry	90	120	180
Filing	140	110	100

Find the sequence that minimizes the total time required to complete the entire task. Also find the total elapsed time and idle times for each operation.

5) Find the sequence that minimizes the total elapsed time to complete the following jobs in the order AB. Find the total elapsed time and idle times for both the machines.

Job	I	II	III	IV	V	VI	VII
Machine A	7	16	19	10	14	15	5
Machine B	12	14	14	10	16	5	7

V) Activity based problems :

Assignment problems :

1) Given below the costs of assigning 3 workers to 3 jobs. Find all possible assignments by trial and error method.

	Jobs		
Workers	X	Y	Z
A	11	16	21
B	20	13	17
C	13	15	12

Among these assignments, find the optimal assignment that minimizes the total cost.

2) Construct a 3 × 3 cost matrix by taking the costs as the first 9 natural numbers and arranging them row wise in ascending order. Find all possible assignments that will minimize the total sum.

3) Given below the costs (in hundred rupees) of assigning 3 operators to 3 different machines. Find the assignment that will minimize the total cost. Also find the minimum cost.

Operators	Machines		
	I	II	III
A	$3i + 4j$	$2j^2 + 5i$	$5j + 3i$
B	$i^3 + 8j$	$7i + j^2$	$4i + j$
C	$2i - 1$	$3i + 5j$	$i^3 + 4j$

Where, i stands for number of rows and j stands for number of columns.

Sequencing problems :

4) Let there be five jobs I, I1, III, IV and V to be processed on two machines A and B in the order AB. Take the first 5 composite numbers as the processing times on machine A for jobs I. II. III, IV, V respectively and the first five odd numbers as the processing times on machine B for jobs V, IV, III, II, I respectively. Find the sequence that minimizes the total elapsed time. Also find the total elapsed time and idle times on both the machines.

5) Consider 4 jobs to be processed on 3 machines A, B and C on the order ABC. Assign processing times to jobs and find the optimal sequence that minimizes the total processing time. Also find the elapsed time.

■■■

8. Probability Distribution

Random Variable (r.v.) :

A random variable is a real valued function defined on the sample space of a random experiment. In other words, the domain random experiment, while its co-domain is thereal line.

There are two types of random variables, namely discrete and continuous

1) **Discrete Random Variable :** A random variable is a discrete random variable if its possible values form a countable set, which may be finite or infinite. The values of a discrete random variable are usually denoted by non-negative integers eg. the number of children in a family, the number of patients in a hospital ward, etc.

2) **Continuous Random Variable :** A random variable is a continuous random variable if its possible values form an interval of real numbers. A continuous random variable has uncountably infinite possible values and these values form an interval of real numbers. eg. heights of trees in a forest, weights of students in a class, daily temperature of a city, speed of a vehicle, and so on.

Probability Distribution of a Discrete Random Variable : It Let the possible values of X be denoted by $x1$, $x2$,$x3$, ..., and the corresponding probabilities be denoted by $p1$, $p2$, $p3$,... Then, the set of ordered pairs $\{(x1, p1), (x2, p2), (x3, p3),...\}$ is called the probability distribution of the random variable X.

Probability Mass Function (p.m.f.) : If there is a function f such that $f\ xi) = pi = P$ $[X = xi]$, for all possible values of X, then f is called the probability mass function (p.m.f.) of x.

$0 \leq pi \leq 1$, $i = 1, 2, 3 \ldots .n$; $\Sigma\ pi = 1$.

Cumulative Distribution Function (c.d.f.) : The cumulative distribution function (c. d. f.) of a discrete random variable X is denoted by F and is defined as follows.

$$F(x) = P[X \leq x] = \sum_{xi \leq x} p[X = x_i] = \sum_{xi \leq x} f(xi)$$

Expected value (Mean) and variance of a random variable :

1) **Expected value (Mean) :** Let X be a random variable whose possible values $x1$, $x2$, $x3$, ..., xn occur with probabilities p_1, p_2, p_3, ... pn respectively. The expected value or arithmetic mean of X, denoted by $E(X)$ or μ is defined by

$$\mu = E(X) = (x_1\ p_1 + x_2\ p_2 + \ldots\ldots\ldots\ldots + x_n p_n) = \sum_{i=1}^{n} x_i p_i$$

2) **Variance :** Let X be a random variable whose possible values $x1,x2,...,xn$ occur with probabilities $p1$, $p2$, $p3$, ... , pn respectively. The variance of X, denot by var (X) or σ^2 is defined as $\sigma^2 = var\ (X) = \sum_{i=1}^{n} x^2 i\ pi - \left(\sum_{i=1}^{n} x^2 i\ pi\right)^2$ OR $var\ (X) = E(X^2) - [\ E(X)]^2$

3) **Standard Deviation :** The non-negative square root of Var (X) is called the standard deviation of the random variable X. That is, $= \sqrt{Var\ (x)}$.

Probability Distribution of a Continuous Random Variable :

- **Probability Density Function (p. d. f.) :**

  Let X be a continuous random variable with the interval (a, b) as its support. The probability density function (p. d. f.) of X is an integrable function f that satisfies the following conditions.

  1) $f\ (n) >, 0$ for all $x \in (a, b)$.

  2) $\int_{b}^{a} f\ (x)\ dx = 1$

- **Cumulative Distribution Functions (c.d.f) :**

  The cumulative distribution function (c. d. f.) of a continuous random variable X or (a, b) is denoted by F and defined by

  $F\ (x) = 0$ for all $(x \leq a)$

  $F\ (x) = \int_{-1}^{x} f\ (x)\ dx = 1$ for $a < n < b$

- **Expected value (Mean) and variance of a random variable :**

  Mean $= F\ (x) = \int_{-\infty}^{\infty} x\ f\ (x)\ dx$, $E\ (x^2) = \int_{-\infty}^{\infty} x^2 f\ (x)\ dx$, and $var\ (X) = E(X^2) - [\ E(X)]^2$

- **Binomial Distribution :**

  A discrete random variable X is said to follow a Binomial distribution with parameter n and p if its probability mass function is given by

  $p\ [X = x] = P\ (x) = {}^{n}C_{x}\ p^{n}\ q^{n-x}$, $x = 0, 1, 2...$n;

  $0 < p < 1$; $q = 1 -$ b

  = 0, otherwise

- **Mean and Variance of Binomial Distribution :**

  Let $X \sim B\ (n, p)$. The mean or expected value of X is denoted by μ. It is also called expected value of X and is denoted by $E\ (X)$ and given $\mu = \boldsymbol{var\ (X) = np}$. The variance is denoted by Var (X) and is given by $\boldsymbol{\sigma^2 = var\ (X) = npq}$.

Solved Examples :

1) The eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

Solution : Let X denote the number of defective eggs in the 10 eggs drawn. Since the drawing is done with replacement, the trials are Bernoulli trials.

Probability of success $= \frac{1}{10}$

$p = \frac{1}{10}$, $q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$ and $n = 10$

$X \sim B\ (n, p)\ p\ [X = x] = p\ (x) = C_x^n\ p^x\ q^{n-x}$

$X \sim B\left(10, \frac{1}{10}\right), p\ [X = x] = p\ (x) = C_x^{10} \left(\frac{1}{10}\right)^x \left(\frac{9}{10}\right)^{10-x}$

Here, $X \geq 1$, $p\ (x \geq 1) = 1 - p\ (x < 1) = 1 - p\ (x = 0)$

$= 1 - C_x^{10} \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{10-0}$

$= 1 - \left(\frac{9}{10}\right)^{10}$

2) If $E\ (x) = 6$ and $Var\ (X) = 4.2$, find n and p.

Solution : $E\ (x) = 6$. i.e. $np = 6$, $var\ (X) = 4.2$. i.e. $npq = 4.2$.

$\frac{npq}{np} = \frac{4.2}{6} = 0.7$, $q = 0.7$

$\therefore\ q = 0.7$,

$p = 1 - q\ p = 1 - 0.7 = 0.3$

$np = 6$

$\therefore\ n \times 0.3 = 6\ \therefore\ n = \frac{6}{0.3} = 20$

Poisson Distribution :

A discrete random variable X is said to have the Poisson distribution with parameter m > 0, if its p.m. is given by $X \sim p\ (m)$, p $(X = x) = \frac{(e^{-m}\ m^x)}{x!}$, $x = 0, 1, 2$ ......, $m = np > 0$

$= 0$, otherwise

Note : 1) For the poision distribution Mean $=$ $E\ (X) =$ and Variance $= var\ (X) = m$. i.e. Mean $=$ Variance $= m$

2) When n is very large and p is very small in the binomial distribution, then X follows the Poisson distribution with parameter $m = np$.

Solved Examples :

1) If $X \sim p\ (m)$ with $m = 5$ and $e^{-5} = 0.00067$, then find (i) $p\ (X = 5)$, (ii) $p\ (X \geq 2)$.

Solution : $X \sim p\ (m),\ p\ (X = x) = \dfrac{(e^{-m}\ m^{x})}{x!}$, $x = 0, 1, 2 ....,$

i) $p\ (X = 5)$, here $m = 5$ and $x = 5$

$$p\ (X = 5) = \frac{(e^{-5}\ 5^{5})}{5!}\ \frac{0.006 \times 3125}{120} = \mathbf{0.1745}$$

ii) $p\ (X \geq 2) = 1 - p\ (X < 2) = 1 - [p\ (X = 0) + p\ \ (X = 1)]$

$$= 1 - \left[\frac{(e^{-5}\ m^{0})}{0!} + \frac{(e^{-5}\ m^{0})}{0!}\right]$$

$$= 1 - \left[\frac{0.0067 \times 1}{1} + \frac{0.0067 \times 5}{1}\right]$$

$= 1 - [0.0067 + 0.0335]$

$= 1 - [0.0402]$

$= \mathbf{0.9598}$

2) In a town, 10 accidents takes place in the span of 50 days. Assuming that the number of accidents follows Poisson distribution, find the probability that there will be 3 or more accidents on a day. (given that $e^{-0.2} = 0.8187$)

Solution : Here $m = \dfrac{10}{50} = 0.2$ and

$X \sim p\ (m)$, with $m = 0.2$ The p. m. of X is

$$p\ (X = x) = \frac{(e^{-m}\ 5^{x})}{x!}\quad x = 0, 1, 2 ....,$$

$p\ (X \geq x) = 1 - p\ (X < x) = 1 - [p\ (X = 0) + p\ (X = 1) + p\ (X = 2)]$

$$= 1 - \left[\frac{(e^{-0.2}(0.2)^{0})}{0!} + \frac{(e^{-0.2}(0.2)^{1})}{1!} + \frac{(e^{-0.2}(0.2)^{2})}{2!}\right]$$

$$= 1 - \left[\frac{0.8187 \times 1}{1} + \frac{0.8187 \times 0.2}{1!} + \frac{0.8187 \times 0.004}{2}\right]$$

$= 1 - [0.8187 + 0.16374 + 0.016374]$

$= 1 - 0.9988$

$= 0.0012$

PROBLEMS FOR PRACTICE

I) Choose the correct options from the given alternatives.

1) The expected value of the sum of two numbers obtained when two fair dice are rolled is ....

   a) 5   b) 6   c) 7   d) 8

2) Given p.d.f. of a continuous r.v. X as $f(x) = \frac{x^2}{3}$ for $-1 < x < 2$ and $f(x) = 0$ otherwise, then $F(1) =$

   a) 19   b) 29   c) 39   d) 49

3) If $E(x) = m$ and $Var(x) = m$ then X follows ....

   a) Binomial distribution   b) Poisson distribution

   c) Normal distribution   d) none of the above

4) The following function represent the p.d.f. of a r. v. X. $f(x) = kx$, for $0 < x < 2$, $f(x) = 0$, otherwise, then the value of k is …

   a) 32   b) 12   c) 1   d) 0

5) If $X \sim B(20, \frac{1}{10})$ then $Var(x) =$ ....

   a) 95   b) 2   c) 59   d) 12

II) Fill in the blanks.

1) If $F(x)$ is distribution function of discrete r.v. X with p.m.f. $P(x) = k\ 4C_x$ for $x = 0$, 1, 2, 3, 4, and $P(x) = 0$ otherwise then $F(-1) =$ ....
2) In Binomial distribution probability of success ...... from trial to trial.
3) In Binomial distribution if n is very large and probability success of p is very small such that $np = m$ (constant) then .... distribution is applied.

III) True or False.

1) If $f(x) = k\,x\,(1-x)$ for $0 < x < 1$ and $f(x) = 0$ otherwise, then k = 12
2) If $X \sim B(n, p)$and $n = 6$ and $p(X = 4) = p(X = 2)$ then $= \frac{1}{2}$.
3) If $P(X = x) = k^4\,C_x$, for $x = 0, 1, 2, 3, 4$, then $F(5) = \frac{1}{4}$ when $F(x)$ is c.d.f.

IV) Solve the following.

1) A sample of 4 bulbs is drawn at random with replacement from a lot of 30 bulbs which includes 6 defectives bulbs. Find the probability distribution of the number of defective bulbs.

2) A random variable X has the following probability distribution :

X	1	2	3	4	5	6	7
$p(x)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Determine (i) k, (ii) $P(X < 3)$, (iii) $P(0 < X < 3)$, (iv) P (X >

3) 70% of the members favour and 30% oppose a proposal in a meeting. The random variable X takes the value 0 if a member opposes the proposal and the value 1 if a member is in favour. Find $E(X)$ and $Var(X)$.

4) Let X be the amount of time for which a book is taken out of library by a randomly selected student and suppose that X has p. d. f.

$f(x) = 0.5x$, for $0 \leq x \leq 5$,

$f(x) = 0$, otherwise.

Calculate (i) $p(x \leq 1)$, (ii) $p(0.5 \leq X \leq 1.5)$, (iii) $p(x \geq 1.5)$.

5) If a r. v. X has p. d. f. $f(x) = \dfrac{c}{x}$ for $1 < x < 3$, $c > 0$

$f(x) = 0$, otherwise.

Find c, E (X), and $Var(X)$. Also find $F(x)$.

6) There are 10% defective items in a large bulk of items. What is the probability that a sample of 4 items will include not more than one defective item?

7) 10 balls are marked with digits 0 to 9. If four balls are selected with replacement. What is the probability that none is marked 0?

8) In a multiple choice test with three possible answers for each of the five questions, what is the probability of a candidate getting four or more correct answers by random choice?

9) If X has Poisson distribution with parameter m and $p(X = 2) = (X = 3)$, then find $P(X \geq 2)$. Use $e^{-3} = 0.0497$.

10) Defects on plywood sheet occur at random with the average of one defect per 50 sq. ft. Find the probability that such a sheet has (i) no defect, (ii) at least one defect. [Use $e^{-3} = 0.3678$.]

11) A player tosses two coins. He wins Rs. 10 if 2 heads appear, Rs. 5 if 1 head appears, and Rs. 2 if no head appears. Find the expected value and variance of winning amount.

12) The probability that a bomb will hit the target is 0.8. Find the probability that, out of 5 bombs, exactly 2 will miss the target.

V) Activity :

1) If X follows Poisson distribution such that $P\ (X = 1) = 0.4$ and $P\ (X = 2) = 0.2$. Complete the following activity to find the variance of X. Also find the probability that x takes the value at most 2.

Here $X \sim p\ (m)$

$\therefore\ P\ [(X = x)] = [.....]$

$P\ [X = 1] = 0.4$

$$\therefore\ 0.4 = \frac{(e^{-m}\ m^1)}{1!}$$

$$\therefore\ e^{-m} = \frac{0.4}{m}$$

Now, given $P\ [X = 2] = 0.2$

$$\therefore\ 0.2 = \frac{(e^{-m}\ m^1)}{2!}$$

$$\therefore\ 0.2 = \frac{\frac{0.4}{m} \times \text{m}^2}{2}$$

$\therefore\ m = [......]$, Varaince = [........]

Now P (X $\leq$ 2) = P (X = 0) + P (X = 1) + P (X = 1)

$$= \frac{e^{-1}\ 1^2}{1!} + 0.4 + 0.2$$

$= \boxed{}$

2) In a large school, 80% of the students like mathematics. A visitor asks each of 4 students, selected at random, Complete the following activity to find the probability that (i) all students like mathematics. (ii) at least 3 students like mathematics. Here n = 4 80% of the students like mathematics

$\therefore$ probability that a student like mathematics is $p = \boxed{[.....]}$

$\therefore\ q = \boxed{[.....]}$

Now, $X \sim$ B (n, p)

For binomial distribution $p\ [X = x] = P\ (x) = C^n_x\ P^x\ q^{n-x}$

(i) all students like mathematics

$P\ [X = x] = \boxed{[.....]}$

(ii) at least 3 students like mathematics.

$P\ [X \geq 3] = P\ [X = 3] + P\ [X = 4]$

$= \boxed{[.....]}$

■■■

Answer Key (Part II)

1. Commission, Brokerage and Discount

I) Multiple Choice Questions.

1) b) del credere agent
2) a) factor
3) c) nominal due date
4) a) The legal due date
5) a) face value
6) b) present worth
7) d) true discount
8) c) list price
9) b) invoice price = Net selling price
10) b) true discount

II) Fill in the blanks.

1) Drawer
2) Auctioneer
3) Catalogue/list
4) commercial discount
5) higher
6) Bankers gain
7) Legal due date
8) A broker
9) Trade, catalogue/list

III) True and False.

1) False
2) False
3) True
4) False
5) True
6) False
7) False
8) True

Exercise 1.1

1) Rs. 760, Rs. 42,340
2) 22
3) Rs. 5,800, 6%
4) 11.75%
5) $r = 20\%$

Exercise 1.2

1) Rs. 10,500
2) Rs. 8,333.33
3) T.D. = Rs. 182.60; BD = Rs. 190.80; BG = Rs. 8.20

4) 15^{th} April 1998

5) 5 Months

6) BG = Rs. 10; TD = Rs. 1000

Acivity

1) Rs. 750 ;

$x - 10,000;$

$\frac{5}{100} - 500;$

$x - 750 - \frac{5x}{100} - 500 = 33,950$

$\frac{9x}{100} = 33,950 + 1250$

$\frac{95x}{100} = 34,200$

$x = \text{Rs. } 36,000$

2) Nominal due date = 19^{th} September 2018

APR 30

May 31

Jul 31

Sep 22

$BD = \frac{sd \times n \times r}{100} = 4015 - 181.3$

2. Insurance and Annuity

I) Multiple Choice Questions.

1) d) All the three

2) d) Converts the possibility of large loss to certainty of a small one

3) c) Present value of annuity

4) b) Perpetuity

5) d) Chance of high Longevity

II) Fill in the blanks.

1) Premium　　2) Property Value

3) Installment　　4) Perpetuity

5) Immediate annuity or Ordinary annuity

III) State whether each of the following is True or False.

1) False　　2) True

3) True　　4) False

5) True

IV) Solve the following problems.

1) i) Rs. 75,000 , ii) Rs. 60,000　　2) Rs. 2,25,000

3) Rs. 1,40,000　　4) Rs. 23,205

5) Rs. 91,120　　6) Rs. 3,641

7) Rs. 31,944

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3. Linear regression

A) 1) c　2) c　3) b　4) c　5) c　6) a　7) d　8) c,　9) b　10) b

11) a　12) c　13) d　14) c　15) b.

B) 1) – 0.4　2) -1　3) $2\,|r|$　4) b_{UV}　5) 0.4

6) 0.625　7) $\frac{6}{5}$　8) 5　9) $-\frac{3}{2}$　10) $\frac{7}{5}$

11) 0.48　12) $y = 30$　13) 13　14) 14.4　15) 9

C) 1) False　2) False　3) True　4) False　5) True

6) False　7) True　8) False　9) False　10) False

11) False　12) False　13) False　14) True　15) True.

D) 1) $X = 0.75\, Y + 2, X = 11,$

2) $Y = -\,0.21X + 136.67,\ Y = 92.57,$

3) $Y = 0.91\,X - 0.97\ ,\ Y = 17.23,$

4) $Y = 3.2\,X - 58,\ X = 0.2Y - 8,\ X = 14,\ Y = 110$

5) $r = \frac{1}{3}$, $Var(X) = 9$

6) $Y = -0.6X + 25.4$, $X = -0.216Y + 17.672$, $Y = 13.4$, $X = 12.272$

7) Mean of X and $Y = (1, 1)$, $r = -\sqrt{\frac{14}{15}}$

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4. Time Series

I) 1) d 2) a 3) a 4) a 5) c

II) 1) Trend 2) Seasonal 3) Least square 4) Graphical 5) Moving average

III) 1) F 2) T 3) F 4) F 5) F

IV) 1) $y = 3.08 + 0.25x$

2)

1971	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982
-	-	1.25	1.75	2.375	3.25	4	4.125	4	4.5	-	-

3) $y = 6 + 0.7x$, $y = 12.3$

4)

1976	1977	1978	1979	1980	1981	1982	1983	1984	1985	1986
-	3.33	3.33	4	6	7	8	6	7.67	8	-

5) $y = 5 + 0.8x$, $y = 9.8$

6)

1962	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972	1973	1974	1975	1976
-	-	0.8	1.4	2.2	3	4	5.2	6.4	7.4	8	8.6	9	-	-

V) 1) Middle year is 1995

a = 4.2857, b = −2

The equation of trend line is $y = 4.2857 - 2x$.

2) 3 yearly moving total = 22, 4.

3 yearly moving average (trend value) = 5.33, 2.67.

3) $n = 10$, two middle years are 2010 and 2011 and $h = 2$

$a = 17.7$ and $b = 0.1$

The equation of trend line is y = 17.7 + 0.1x

Put $x = 11$ then $y = 18.8$

4) 4 yearly moving total = - , 66, 68

4 yearly centered total = 136, 135

4 yearly centered moving average (trend values) = - , 16.625, 16.875

---

5. Index Number

Q.1) A) 1) a 2) c 3) d 4) c v) b

B) i) False ii) False iii) False

C) i) $p_{01}(\text{F}) = \sqrt{\dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times \dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0}} \times 100$ ii) arithmetic mean

iii) 20 iv) 129 v) Current period

Q. 2) 144.74

Q.3) 117.14

Q. 4) 121.34

Q.5) $x = 25$

Q. 6) 141.76, 140.53, 141.15, 141.03

Q.7) 166.67, 87.5, 127.08

Q. 8) $p_0(\text{L}) = 8, p_{01} = 2$

Q.9) $p_{01}(\text{D - B}) = p_{01}(\text{F})$

Q.10) CLI = 160

Q.11) 77

Q.12) $X = 18$

Q.13) $20 + 5x$, $\Sigma p_1 q_1$, 30, $x = 2$

Q.14) 40, Σp_0, $350 + y$, 66

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6. Linear Programming

I) Multiple Choice Questions.

1) d) The vertex which is at maximum distance from (0, 0).

2) c) If LPP has two optimal solutions then it has infinitely many optimal solutions.

3) b) a function to be maximized or minimized.

4) c) $\dfrac{235}{19}$

5) d) (40,15)

6) b) All of the given function

7) b) 13

8) a) (2, 2)

9) c) (0, 0)

10) c) (3, 4)

II) Fill in blanks

1) I 2) Vertex 3) III and IV

4) $x \geq 3$ and $y \geq 2$ 5) $x + y \geq 2$

III) State whether each of the following is True or False.

1) True 2) False 3) True 4) True 5) True

IV) State whether each of the following is True or False.

1) Maximize $P = 350x + 400\,y$ subject to $x \geq 0, y \geq 0$

$3x + 2y \leq 120$ and $2x + 5y \leq 60$

2) Maximize $Z = 500x + 750y$ subject to $x \geq 0, y \geq 0$

$2x + 3y \leq 40$ and $x + 4y \leq 70$

3) Minimize $C = 20x + 6y$ subject to $x >= 0, y >= 0$

$x \geq 4$ and $y \leq 2, x + y \geq 5$

4) $x = 2$, $y = 3$ and Maximize $z = 95$

5) $x = 0, y = 5$ and Minimize $z = 5$

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7. Assignment Problem and Sequencing

I) Multiple Choice Questions.

1) c) Both a and b

2) c) to find the sequence in which jobs on hand are to be processed. to minimize the total time required for processing the jobs

3) b) Hungarian Method.

4) d) The time of passing depends on the order of Machining.

5) a) Converting all profits to opportunity.

6) c) Number of rows is equal to number of columns.

7) a) Dummy column is added.

8) b) A-B-C

9) b) Number of jobs to equal number of persons at minimum cost.

II) Fill in blanks.

1) Number of rows is not equal to the number of columns.

2) Square

3) Zero

4) Total elapsed time

5) Maximum

6) A - D - B - C

III) State whether each of the following is True or False.

1) False 2) False 3) True 4) True 5) False

IV) Solve the following problems.

1) 1-I, 2-III, 3-IV, 4-II, 5-V and Total mileage 39 miles.

2) M1-A, M2-B, M3-E, M4-D and Total Cost Rs. 32

3) Optimal Sequence : 4 - 1 - 3 - 2 - 5 - 6

Idle time for turning operator = 1 min

Total elapsed time = 43 min

Idle time for threading = 6 min

4) Optimal Sequence : 4 – 1 – 3 – 2 – 5 – 6

Idle time for data entry operation = 100 min

Total elapsed time = 490 min

Idle time for filling = 140 min

5) Optimal Sequence : VII – I – VI – V – III – II – VI or

VII – I – VI – V – II – III – VI

Idle time for machine A = 5 units

Total elapsed time = 91 min

Idle time for machine B = 13 units

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8. Probability Distribution

I) 1) c) 7 2) b) $\frac{2}{9}$ 3) b) Poisson Distribution

4) b) $\frac{2}{9}$ 5) a) $\frac{9}{5}$

II) 1) 0 2) Remain constant/independent 3) Poisson.

III) 1) False 2) True 3) False

IV) 1)

X0	0	1	2	3	4
P(X)	$\left(\frac{4}{5}\right)^2$	$4\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)$	$6\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2$	$6\left(\frac{4}{5}\right)^1\left(\frac{1}{5}\right)^3$	$\left(\frac{1}{5}\right)^4$

2) i) $\frac{1}{10}$ ii) $\frac{3}{10}$ iii) $\frac{3}{10}$ iv) $\frac{1}{5}$

3) 0.7, 0.21

4) i) $\frac{1}{4}$ ii) $\frac{1}{2}$ iii) $\frac{7}{16}$

5) i) $\frac{1}{log\ 3}$ ii) $\frac{2}{log\ 3}$ iii) $\frac{4\ [log\ 3 - 1]}{(log\ 3)^2}$ $F(x) = \frac{log\ x}{log\ 3}$ $1 < x < 3$

6) $1.3 \times (0.9)^3 = 0.9477$

7) $\left(\frac{9}{10}\right)^4$

8) $\frac{11}{243}$

9) 0.8012

10) i) 0.3678 ii) 0.6322

11) Rs. 5.5, Rs. 8.25

12) $\frac{128}{625}$

V) 1) $\frac{(e^{-m}m^x)}{x!}$, 1, 1, 0.9678

2) $\frac{4}{5}$, $\frac{1}{5}$, $\frac{256}{625}$, $\frac{512}{625}$

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Std. XII - Subject : Mathematics and Statistics
(Commerce) Part - II

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