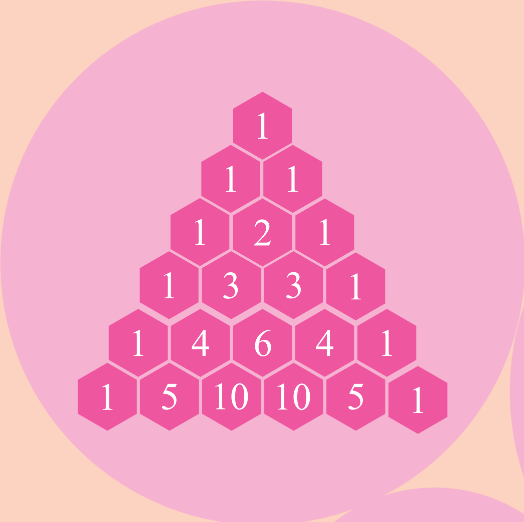
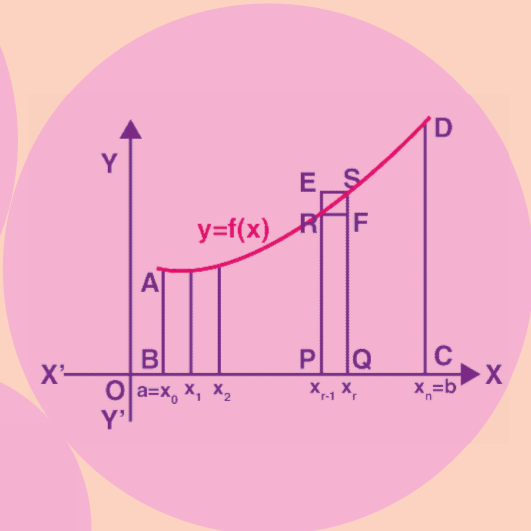
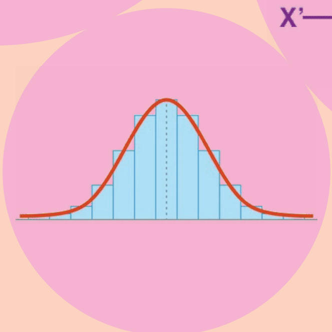
"Comprehensive Support for Students in Mathematics subject seeking to Overcome Past Setbacks."







MATHEMATICS AND STATISTICS

Std. - XII

(Arts and Science)

Part - II

State Council of Educational Research and Training, Maharashtra, Pune

"Comprehensive Support for Students in Mathematics subject seeking to Overcome Past Setbacks."

Std. - XII
Subject - Mathematics and Statistics (Arts and Science)
Part - II

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Std. XII Subject : Mathematics and Statistics (Arts and Science) Part - II

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'Comprehensive Support for Students in Mathematics subject seeking to Overcome Past Setbacks.'

Specialized Mathematics Study Materials for HSC Students

Subject : Mathematics and statistics (Arts and Science) Code : 40

OBJECTIVES OF THE BOOKLET

This booklet is prepared for the help of the students who will be appearing for the Supplementary Examination and thereafter too. It is prepared as such students could not score the minimum score to pass in the written Board examination held in February 2024.

This booklet is designed to boost the confidence of the students. It will definitely help them to score good marks in the forthcoming examination. It will be a great support for the students who lack behind others.

It is prepared in a systematic and easiest way by the expert teachers. The students are aware of the text book as well as the examination pattern (MCQ's, 1 Mark, 2 Marks, 3 Marks and 4 Marks questions). Still, this booklet elaborates every segment in detail. It considers the level of the students.

By studying as suggested in the booklet, we are quite sure that the students will be able to practice a lot with given guidelines. They will score and step into the world of success.

The main objectives can be summarized as under :

1) To facilitate the essential study material to the students to confidently face the HSC Board Examination.
2) To help every low achiever student to achieve 100% success at the HSC Board Examination.
3) To motivate the students to score more than their expectation in the Mathematics Subject which they find as most difficult.
4) To include tools and exercises that allow students to evaluate their own progress and understand their improvement areas.
5) To help the teachers to reach out to students who struggle to pass in the Mathematics subject at the HSC Board Exam with the help of this material.
6) Each chapter in the booklet contains important concepts in short.
7) Based on these concepts simple solved examples are given.
8) Practice questions with hints and answers are given.
9) Two practice question papers will definitely help students.

INTRODUCTION

Dear Students,

It does not matter if you did not score well in the regular examination held in February 2024. Remember, "every setback is a setup for a comeback." Your previous attempt must have taught you something valuable. We believe in your potential to overcome this hurdle and excel in your upcoming exams.

After a comprehensive analysis of the results, SCERT, Maharashtra, Pune has taken an initiative for the upliftment of students who could not achieve the minimum passing score. It was found that some fundamental concepts were not clear to the students. Hence, a significant effort was made to prepare this booklet.

This booklet is designed specifically for those who did not achieve the desired results in their previous Mathematics exam. We understand that facing a setback can be challenging, but it also presents an invaluable opportunity for growth and learning. Our goal with this booklet is to provide you with comprehensive resources and targeted exercises to help you strengthen your understanding of key mathematical concepts. We have carefully curated the content to address common areas of difficulty and to reinforce fundamental principles essential for success in Mathematics.

This booklet will help you to prepare for the supplementary examination. Through a combination of clear explanations, step-by-step problem-solving strategies, and ample practice questions, we aim to build your confidence and competence in the subject. Remember, perseverance and a positive mindset are crucial as you work through this material.

Use this booklet diligently, seek help when needed, and stay committed to your studies. With dedication and effort, you can turn this experience into a stepping stone toward academic success. This resource will also prove to be extremely useful for teachers as they assist students in preparing for the supplementary examination. It will boost your confidence to appear for the exam once again. New students in the coming years can also benefit from this booklet. Best wishes for your journey ahead.

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Part II

1. Differentiation

1.1 Definition :

If $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ exists, then this limits is called is the derivative of $f(x)$ w.r.t.x and is denoted by $f'(x)$ $\therefore (x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

Leibnitz's Notation :

Let $y = f(x)$ stbe a function of x. Let δy be a small change in y corresponding to change δx in x, then $\lim\limits_{h\to 0}\left[\dfrac{\delta y}{\delta x}\right]$ exists,is called the derivative of y w.r.t.x and is denoted by $\dfrac{dy}{dx}$ *i. e.* $\dfrac{dy}{dx} = \lim\limits_{h\to 0}\left[\dfrac{\delta y}{\delta x}\right]$

Let $y = f(x)$ then di ff.y w.r.t. x is $\dfrac{dy}{dx} = \dfrac{d}{dx} f(x) = f'(x)$

Sr. No.	Derivatives of Standard Function	Derivative of Composite Function
1)	$\dfrac{d}{dx}(k) = 0,$ k is const.	
2)	$\dfrac{d}{dx} x^n = nx^{n-1}$	$\dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$
3)	$\dfrac{d}{dx}\dfrac{1}{x} = \dfrac{-1}{x^{2"}}$	$\dfrac{d}{dx}\dfrac{1}{f\sqrt{(x)}} = \dfrac{-1}{[f(x)]^2} f'(x)$
4)	$\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$	$\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{1}{2\sqrt{f(x)}} f'(x)$
5)	$\dfrac{d}{dx}\sin x = \cos x$	$\dfrac{d}{dx}\sin f(x) = \cos f(x) f'(x)$
6)	$\dfrac{d}{dx}\cos x = -\sin x$	$\dfrac{d}{dx}\cos f(x) = -\sin f(x) f'(x)$
7)	$\dfrac{d}{dx}\tan x = \sec^2 x$	$\dfrac{d}{dx}\tan f(x) = \sec^2 f(x) f'(x)$
8)	$\dfrac{d}{dx}\cot x = -\operatorname{cosec}^2 x$	$\dfrac{d}{dx}\cot f(x) = -\operatorname{cosec}^2 f(x) f'(x)$

Sr. No.	Derivatives of Standard Function	Derivative of Composite Function
9)	$\frac{d}{dx} \sec x = \sec x \tan x$	$\frac{d}{dx} \sec f(x) = \sec (x) \tan f(x) f'(x)$
10)	$\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$	$\frac{d}{dx} \operatorname{cosec} f(x)$
11)	$\frac{d}{dx} = a^x \log a$, $a > 1,\ a \neq 1$	$\frac{d}{dx} a^{f(x)} = a^{f(x)} \log a \ \ f'(x), a > 1,\ a \neq 1$
12)	$\frac{d}{dx} \ e^x = e^x$	$\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$
13)	$\frac{d}{dx} \log x, = \frac{1}{x}$	$\frac{d}{dx} \log f\ (x) = \frac{1}{f(x)} \ f'(x)$
14)	$\frac{d}{dx} \log_a x, = \frac{1}{x \log a}$ $a > 1, a \neq 1$	$\frac{d}{dx} \log_a f\ (x) = \frac{1}{f(x) \log a} f'(x)$ $a > 1, a \neq 1$

Rules of Differentiation : Let u and v are differentiable functions of x.

1) If $y = u + v$, then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

2) If $y = u - v$, then $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

3) If $y = u \cdot v$, then $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

4) If $y = k \cdot u$, then $\frac{dy}{dx} = k \cdot \frac{du}{dx}$ where k is constant

5) If $y = u \cdot v \cdot w$, then $\frac{dy}{dx} = uv \frac{dw}{dx} + uw \frac{dv}{dx} + vw \frac{du}{dx}$

6) If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Theorem : If y is differential function of u and u is differential function of x, then y is differential function of x and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Proof : A Let δy and δu be a small change in y and u corresponding to change δx in x

As $\delta x \to 0$, $\delta y \to 0$ and $\delta u \to 0$

y is differential function of u then $\frac{dy}{du} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u}$

u is differential function of x then $\frac{du}{dx} = \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$

Now, $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$

Taking limit as $\delta x \to 0$

$$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \cdot \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

$$= \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \cdot \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \qquad \because \delta x \to 0, \delta u \to 0$$

$$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Here L.H.S. is exists so R.H.S. is exists $\because$ y is differential function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

To finding the derivative of composite function is called **Chain Rule**

1) **Find $\frac{dy}{dx}$ If** $y = e^{\tan x}$

Solution : $y = e^{\tan x}$

Differentiate w.r.t.x

$$\frac{dy}{dx} = e^{\tan x} \frac{d}{dx} \tan x = e^{\tan x} \sec^2 x$$

2) **Find $\frac{dy}{dx}$ If $y = \sqrt{\tan\sqrt{x}}$**

Solution : $y = \sqrt{\tan\sqrt{x}}$

Differentiate w.r.t.x

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan\sqrt{x}}} \frac{d}{dx} \tan\sqrt{x} = \frac{1}{2\sqrt{\tan\sqrt{x}}} \sec^2 \sqrt{x} \frac{d}{dx} \sqrt{x}$$

$$\frac{1}{2\sqrt{\tan\sqrt{x}}} \sec^2 \sqrt{x} \frac{1}{2\sqrt{x}}$$

3) Find $\frac{dy}{dx}$ If $y = \cot^2 (x^3)$

Solution : $y = \cot^2 (x^3)$

Differentiate w.r.t.x

$$\frac{dy}{dx} = 2 \cot (x)^3 \frac{d}{dx} [\cot (x)^3] = 2 \cot (x)^3 [- \text{cosec}^2 (x)^3] \frac{d}{dx} (x)^3$$

$$= 2 \cot(x)^3[-\text{cosec}^2(x^3)](3x^2) = -6x^2\cot(x)^3\text{cosec}^2(x^2)$$

4) Find $\frac{dy}{dx}$ If $y = \log (\sec e^{x2})$

Solution : $y = \log (\sec e^{x2})$ Differentiate w.r.t.x

$$\frac{dy}{dx} = \frac{1}{\sec e^{x2}} \frac{d}{dx} \sec e^{x2} = \frac{1}{\sec e^{x2}} \sec e^{x2} \tan e^{x2} \frac{d}{dx} e^{x2}$$

$$= \tan e^{x2} e^{x2} \frac{d}{dx} x^2 = \tan e^{x2} e^{x2} 2x = 2x\, e^{x2} \tan e^{x2}$$

5) Find $\frac{dy}{dx}$ If $y = \log \left[4^x \left(\frac{x^2+2}{x^2-1}\right)^{\frac{2}{3}}\right]$

Solution : $y = \log \left[4^x \left(\frac{x^2+2}{x^2-1}\right)^{\frac{2}{3}}\right] = \log (4^x) + \log \left(\frac{x^2+1}{x^2-1}\right)^{\frac{2}{3}} = x\log 4 + \frac{2}{3}\log \left(\frac{x^2+2}{x^2-1}\right)$

$$\therefore\ y = x \log 4 + \frac{2}{3} \log (x^2 + 1) - \frac{2}{3} \log (x^2 - 1)$$

Differentiate w.r.t.x

$$\frac{dy}{dx} = \frac{d}{dx} x \log 4 + \frac{2}{3} \frac{d}{dx} \log (x^2 + 1) - \frac{2}{3} \frac{d}{dx} \log (x^2 - 1)$$

$$= \log 4 + \frac{2}{3} \frac{1}{(x^2+1)} 2x - \frac{2}{3} \frac{1}{(x^2-1)} 2x$$

$$= \log 4 + \frac{4x}{3(x^2+1)} - \frac{4x}{3(x^2-1)}$$

$$= \log 4 + \frac{12x^3 - 12x - 12x^3 - 12x}{9(x^4-1)} = \log 4 + \frac{-24x}{9(x^4-1)}$$

$$= \log 4 + \frac{8x}{3(x^4-1)}$$

6) Find $\frac{dy}{dx}$ If $y = \frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}$

Solution : $y = \frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}$

Differentiate w.r.t.x

$$\frac{dy}{dx} = \frac{d}{dx}\frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}$$

$$= \frac{(e^{\sqrt{x}}-1)\frac{d}{dx}(e^{\sqrt{x}}+1) - (e^{\sqrt{x}}+1)\frac{d}{dx}(e^{\sqrt{x}}-1)}{(e^{\sqrt{x}}-1)^2}$$

$$= \frac{(e^{\sqrt{x}}-1)\, e^{\sqrt{x}}\frac{1}{2\sqrt{x}} - (e^{\sqrt{x}}+1)\, e^{\sqrt{x}}\frac{1}{2\sqrt{x}}}{(e^{\sqrt{x}}-1)}$$

$$= \frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}(e^{\sqrt{x}}-1) - (e^{\sqrt{x}}-1)}{(e^{\sqrt{x}}-1)^2} = \frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}(-2)}{(e^{\sqrt{x}}-1)^2} = \frac{-e^{\sqrt{x}}}{\sqrt{x}\,(e^{\sqrt{x}}-1)^2}$$

Exercise 1.1

Differentiate the following w. r. t. *x*.

1) $y = \sqrt{x^2+5}$ 2) $y = \sin(\log x)$ 3) $\log x^5 + 4$

4) $5^{3\cos x - 2}$ 5) $y = \sqrt{\sin x^3}$ 6) $y = \log[\cos(x^5)]$

7) $y = \log\left[e^{3x}.\frac{(3x+4)^{\frac{2}{3}}}{\sqrt[3]{2x+5}}\right]$ 8) $y = \log\left[\sqrt{\frac{1-\cos\left(\frac{3x}{2}\right)}{1+\cos\left(\frac{3x}{2}\right)}}\right]$

Derivative of Inverse Functions :

Theorem : If $y = f(x)$ is a derivable function of x such that the inverse function $x = f^{(-1)}(y)$ is defined, then show that $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$ where $\left(\frac{dy}{dx}\right) \neq 0$

Proof : Let δy and δx be a small increments in x and y resp.

As $\delta x \to 0$, $\delta y \to 0$ and As $\delta y \to 0$, $\delta x \to 0$

y is differential function of x then $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

Now, $\frac{\delta x}{\delta y} = \frac{1}{\left(\frac{\delta y}{\delta x}\right)}$

Taking limit as $\delta x \to 0$

$$\therefore \lim_{\delta x \to 0} \frac{\delta x}{\delta y} = \frac{\lim_{\delta x \to 0} 1}{\lim_{\delta x \to 0} \frac{\delta y}{\delta x}} = \frac{1}{\lim_{\delta x \to 0} \frac{\delta y}{\delta x}}$$

$$\therefore \lim_{\delta x \to 0} \frac{\delta x}{\delta y} = \frac{1}{\frac{dy}{dx}}$$

$$\therefore \lim_{\delta y \to 0} \frac{\delta x}{\delta y} = \frac{1}{\frac{dy}{dx}} \qquad \because \delta x \to 0, \delta y \to 0$$

Here L.H.S. is exists so R.H.S. is exists

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \text{ where } \frac{dy}{dx} \neq 0 \qquad \text{OR } \frac{dy}{dx} = \frac{1}{\frac{dy}{dx}} \text{ where } \frac{dx}{dy} \neq 0$$

Derivative Of Inverse Trigonometric Functions :

Sr. No	Derivatives of Standard Fun.	Derivative of Composite Function
1)	$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$	$\frac{d}{dx} \sin^{-1} f(x) = \frac{1}{\sqrt{1 - [f(x)]^2}} f'(x)$
2)	$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$	$\frac{d}{dx} \cos^{-1} f(x) = \frac{-1}{\sqrt{1 - [f(x)]^2}} f'(x)$
3)	$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$	$\frac{d}{dx} \tan^{-1} f(x) = \frac{1}{\sqrt{1 + [f(x)]^2}} f'(x)$

Sr. No	Derivatives of Standard Fun.	Derivative of Composite Function
4)	$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$	$\frac{d}{dx}\cot^{-1} f(x) = \frac{-1}{\sqrt{1+[f(x)]^2}} f'(x)$
5)	$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}\sec^{-1} f(x) = \frac{1}{f(x)\sqrt{[f(x)]^2-1}} f'(x)$
6)	$\frac{d}{dx}\operatorname{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}\operatorname{cosec}^{-1} f(x) = \frac{-1}{f(x)\sqrt{[f(x)]^2-1}} f'(x)$

1) Differentiate $\tan^{-1}\sqrt{x}$ w.r.t.x

Solution : Let $y = \tan^{-1}\sqrt{x}$

Differentiate y w.r.t.x

$$\frac{dy}{dx} = \frac{1}{1+\sqrt{(x)}^2}\frac{d}{dx}\sqrt{x} = \frac{1}{1+x}\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}$$

2) Find. $\frac{dy}{dx}$ If $y = \operatorname{cosec}^{-1}\left(\frac{1}{\sin 5^x}\right)$

Solution : $y = \operatorname{cosec}^{-1}\left(\frac{1}{\sin 5^x}\right) = \operatorname{cosec}^{-1}(\operatorname{cosec} 5^x) = 5^x$

Differentiate y w.r.t.x

$$\frac{dy}{dx} = 5^x \log 5$$

3) Find $\frac{dy}{dx}$ If $y = \tan^{-1}\left(\frac{\sin}{1+\cos x}\right)$

Solution : $y = \tan^{-1}\left(\frac{\sin}{1+\cos x}\right)$

$$y = \tan^{-1}\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) = \tan^{-1}(\tan\frac{x}{2}) = \frac{x}{2}$$

Differentiate y w.r.t.x

$$\frac{dy}{dx} = \frac{1}{2}$$

4) Find $\frac{dy}{dx}$ If $y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

Solution : $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

$$y = \tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}} = \tan^{-1}\sqrt{tan^2\frac{x}{2}} = \tan^{-1}\left(tan\frac{x}{2}\right) = \frac{x}{2}$$

Differentiate y w.r.t.x

$$\frac{dy}{dx} = \frac{1}{2}$$

5) Find $\frac{dy}{dx}$ If $y = \cot^{-1}\left(\frac{a\cos x - a\sin x}{a\cos x + a\sin x}\right)$

Solution : $y = \cot^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) = \tan^{-1}\left(\frac{a\cos x + b\sin x}{b\cos x - a\sin x}\right)$

$$= \tan^{-1}\left(\frac{\frac{a}{b} + \tan x}{1 - \frac{a}{b}\tan x}\right) = \tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}(\tan x) = \tan^{-1}\left(\frac{a}{b}\right) + x$$

Differentiate y w.r.t.x

$$\frac{dy}{dx} = 0 + 1 = 1$$

6) Find $\frac{dy}{dx}$ If $y = \tan^{-1}\left(\frac{\cos 7x}{1+\sin 7x}\right)$

Solution : $y = \tan^{-1}\left(\frac{\cos 7x}{1+\sin 7x}\right)$

$$= \tan^{-1}\left(\frac{\cos^2\frac{7x}{2} - \sin^2\frac{7x}{2}}{\left(\cos\frac{7x}{2} + \sin\frac{7x}{2}\right)^2}\right)$$

$$= \tan^{-1}\left(\frac{\left(\cos\frac{7x}{2} - \sin\frac{7x}{2}\right)\left(\cos\frac{7x}{2} + \sin\frac{7x}{2}\right)}{\left(\cos\frac{7x}{2} - \sin\frac{7x}{2}\right)^2}\right)$$

$$= \tan^{-1}\left(\frac{\cos\frac{7x}{2} + \sin^2\frac{7x}{2}}{\left(\cos\frac{7x}{2} - \sin\frac{7x}{2}\right)}\right) = \tan^{-1}\left(\frac{1 + \tan\frac{7x}{2}}{1 - \tan\frac{7x}{2}}\right)$$

$$= \tan^{-1}(1) + \tan^{-1}\left(\tan\frac{7x}{2}\right)$$

$y = \frac{\pi}{2} + \frac{7x}{2}$ Differentiate y w.r.t.x

$$\frac{dy}{dx} = \frac{7}{2}$$

7) Find $\frac{dy}{dx}$ If $y = \tan^{-1}\left(\frac{8x}{1 - 15x^2}\right)$

Solution : $y = \tan^{-1}\left(\frac{8x}{1 - 15x^2}\right) = \tan^{-1}\left(\frac{5x + 3x}{1 - (5x)(3x)}\right) = \tan^{-1}(5x) + \tan^{-1}(3x)$

Differentiate y w.r.t.x

$$\frac{dy}{dx} = \frac{1}{1 + (5x)^2}\frac{dy}{dx}(5x) + \frac{1}{1 + (3x)^2}\frac{dy}{dx}(3x) = \frac{5}{1 + 25x^2} + \frac{3}{1 + 9x^2}$$

8) Find $\frac{dy}{dx}$ If $y = \cos^{-1}\left(\frac{3\cos 3x - 4\sin 4x}{5}\right)$

Solution : $y = \cos^{-1}\left(\frac{3\cos 3x - 4\sin 4x}{5}\right) = \cos^{-1}\left(\frac{3}{5}\cos 3x - \frac{4}{5}\sin 3x\right)$

Here. $\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{5} + \frac{16}{5} = 1$

Put $\frac{3}{5} = \cos\alpha$ and $\frac{4}{5} = \sin\alpha$; Where α ; is constant

$y = \cos^{-1}(\cos\alpha \cos 3x - \sin\alpha \sin 3x) = \cos^{-1}(\cos(\alpha + 3x) = \alpha + 3x$

Differentiate y w.r.t.x

$$\frac{dy}{dx} = 3$$

Derivative Of Inverse Trigonometric Functions by Substitution :

Following are some substitution for expression in finding derivatives.

i) $\sqrt{a^2 - x^2}$ OR $a^2 - x^2$ put $x = a\sin\theta$ OR $x = a\cos\theta$

ii) $\sqrt{a^2 + x^2}$ OR $a^2 + x^2$ put $x = a \tan\theta$ OR $x = a \cot\theta$

iii) $\sqrt{x^2 - a^2}$ OR $x^2 - a^2$ put $x = a \sec\theta$ OR $x = a$ cosscθ

iv) $\sqrt{a - x}$ OR $\sqrt{a + x}$ put $x = a \cos\theta$

v) $\sqrt{\frac{a - x}{a + x}}$ OR $\sqrt{\frac{a + x}{a - x}}$ put $x = a \cos\theta$

vi) $\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$ OR $\sqrt{\frac{a^2 + x^2}{a^2 - x^2}}$ put $x^2 = a^2 \cos\theta$

Some more substitution for expression in finding derivatives.

i) $\frac{2x}{1 + x^2}$ put $x = \tan\theta$ we get $\frac{2x}{1 + x^2} = \sin 2\theta$

ii) $1 - 2x^2$ put $x = \sin\theta$ we get $1 - 2x^2 = \cos 2\theta$

iii) $2x^2 - 1$ put $x = \cos\theta$ we get $2x^2 - 1 = \cos 2\theta$

iv) $\frac{1 - x^2}{1 + x^2}$ put $x = \tan\theta$ we get $\frac{1 - x^2}{1 + x^2} = \cos 2\theta$

v) $\frac{2x}{1 - x^2}$ put $x = \tan\theta$ we get $\frac{2x}{1 - x^2} = \tan 2\theta$

vi) $3x - 4x^3$ put $x = \sin\theta$ we get $3x^2 - 4x^3 = \sin 3\theta$

vii) $4x^3 - 3x$ put $x = \cos\theta$ we get $4x^3 - 3x = \cos 3\theta$

viii) $\frac{3x - x^3}{1 + 3x^2}$ put $x = \tan\theta$ we get $\frac{3x - x^3}{1 + 3x^2} = \tan 3\theta$

9) Find $\frac{dy}{dx}$ If $y = \sin^{-1} 2x\sqrt{1 - x^2}$

Solution : $y = \sin^{-1} 2x\sqrt{1 - x^2}$

put $x = \sin\theta \therefore \theta = \sin^{-1} x$

$y = \sin^{-1}\left(2 \sin\theta \sqrt{1 - \sin^2\theta}\right) = \sin^{-1}\left(2 \sin\theta \sqrt{\cos^2\theta}\right)$

$= \sin^{-1} (2 \sin\theta \cos\theta)$

$y = \sin^{-1} (\sin 2\theta) = 2\theta = 2\sin^{-1} x$

Differentiate y w.r.t.x

$\frac{dy}{dx} = \frac{2}{\sqrt{1 - x^2}}$

Exercise 1.2

Differentiate y w.r.t.x

1) $\sin{-1}(x^3)$ 2) $\cot{-1}\left(\frac{1}{x^2}\right)$ 3) $\sin^2(\sin{-1}(x^2))$

4) $\cos^{-1}(4\cos^3 x - 3\cos x)$ 5) $\sin^{-1}\left(\sqrt{\frac{1-\cos x}{2}}\right)$ 6) $\tan^{-1}\left(\frac{1-\cos}{\sin}\right)$

7) $\cot^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ 8) $\sin^{-1}\frac{2\cos x + 3\sin}{\sqrt{13}}$ 9) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

10) $\operatorname{cosec}^{-1}\left(\frac{1}{3x-4x^2}\right)$ 11) $\tan^{-1}\left(\frac{2e^x}{1-e^{2x}}\right)$ 12) $\cos^{-1}\left(\frac{1-9x^2}{1+9x^2}\right)$

13) $\tan^{-1}\left(\frac{4x}{1+21\ ^2}\right)$ 14) $\tan^{-1}\left(\frac{7x}{1-21x^2}\right)$

Logarithmic Differentiation :

A function in the form function to the power function i.e. $y = (f(x))^{g(x)}$ then taking log on both sides, and solve the example : $y = (f(x))^{g(x)}$

taking log on both sides, we get $\log y = \log (f(x))^{g(x)}$ $\log y = g(x) \log f(x)$

Now Differentiate y w.r.t.x

1) If $y = x^{\tan-1} x$, then $\frac{dy}{dx}$

Solution : $y = x^{\tan-1} x$

taking log on both sides, we get

$\log y = \log x^{\tan-1} x = \tan^{-1} x \log x$

Differentiate y w.r.t.x

$$\frac{1}{y}\frac{dy}{dx} = \tan^{-1} x \frac{d}{dx} = \log x + \log x \frac{d}{dx} \tan^{-1} x$$

$$\frac{dy}{dx} = y\left(\tan^{-1} x \frac{1}{x} + \log x \frac{1}{1+x^2}\right) = x \tan^{-1} x \left(\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2}\right)$$

2) If $y = x^{x^x}$ then find $\frac{dy}{dx}$

Solution : $y = x^{x^x}$

taking log on both sides, we get

$\log y = \log x^{x^x} = x^x \log x$

Differentiate y w.r.t.x

Now $u = x^x$

$\log u = x \log x$

Differentiate y w.r.t.x

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log x + \log x\frac{d}{dx}x$$

$$\frac{du}{dx} = u\left(x\frac{1}{x} + \log x\right)$$

$$\frac{d}{dx}x^x = x^x(1+\log x)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = x^x\frac{1}{x} + \log x\ x^x(1+\log x)$$

$$\frac{dy}{dx} = y\ x^x\left[\frac{1}{x} + \log x\ (1+\log x)\right] = x^{x^x}\ x^x\left[\frac{1}{x} + \log x\ (1+\log x)\right]$$

2) If $y = e^{\tan x} + (\log x)^{\tan x}$ then find $\frac{dy}{dx}$

Solution : $y = e^{\tan x} + (\log x)^{\tan x}$

Put $u = (\log x)^{\tan x}$

$y = e^{\tan x} + u$

Differentiate w.r.t.x

$$\frac{dy}{dx} = e^{\tan x}\frac{d}{dx}\tan x + \quad \frac{du}{dx} = e^{\tan x}\sec^2 x + \quad \frac{du}{dx}$$

Now, $u = (\log x)^{\tan x}$

$\therefore \log u = \log (\log x)^{\tan x} = \tan x \log (\log x)$

Differentiate w.r.t.x

$$\frac{1}{x}\frac{du}{dx} = \tan x\ \frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}\tan x$$

$$\frac{du}{dx} = u\left(\tan x\frac{1}{\log x}\frac{1}{x} + \log(\log x)\sec^2 x\right)$$

$$= (\log x)^{\tan x}\left(\frac{\tan x}{x\log x} + \log(\log x)\sec^2 x\right)$$

$$\therefore \frac{dy}{dx} = e^{\tan x}\sec^2 x + (\log x)^{\tan x}\left(\frac{\tan x}{x\log x} + \log(\log x)\sec^2 x\right)$$

Exercise 1.3

Differentiating the following w.r.t.x

1) $\dfrac{e^{x^2}(\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}}\cos^3 x}$ 2) $x^a + x^x + a^x$ 3) $(\sin x)^{\tan x} - x^{\log x}$

Derivative of Implicit Functions :

Generally, we write the function in the form y=f(x), it is called explicit function. If functions involving relationship between x and y, which can not be written in explicit form i.e. can not separate variable x and y are called implicit functions.

Solved Examples :

1) If $x\sqrt{xy} + y = 1$, then find $\dfrac{dy}{dx}$

Solution : $x + \sqrt{xy} + y = 1$

Differentiating the following w.r t.x

$$\frac{d}{dx}x + \frac{d}{dx}\sqrt{xy} + \frac{d}{dx}y = 0$$

$$1 + \frac{1}{2\sqrt{xy}}\frac{d}{dx}xy + \frac{dy}{dx} = 0$$

$$1 + \frac{1}{2\sqrt{xy}}\left(x\frac{dy}{dx} + y\right) + \frac{dy}{dx} = 0$$

$$1 + \frac{1}{2\sqrt{xy}}\frac{d}{dx} + \frac{1}{2\sqrt{xy}} + \frac{dy}{dx} = 0$$

$$\left(\frac{x}{2\sqrt{xy}} + 1\right)\frac{dy}{dx} = -1 - \frac{y}{2\sqrt{xy}}$$

$$\left(\frac{x + 2\sqrt{xy}}{2\sqrt{xy}}\right)\frac{dy}{dx} = \left(\frac{-2\sqrt{xy} - y}{2\sqrt{xy}}\right)$$

$$\frac{dy}{dx} = \left(\frac{-2\sqrt{xy} - y}{2\sqrt{xy}}\right)\left(\frac{2\sqrt{xy}}{x + 2\sqrt{xy}}\right)$$

$$\frac{dy}{dx} = \left(\frac{-2\sqrt{xy} - y}{x + 2\sqrt{xy}}\right) = -\left(\frac{y + 2\sqrt{xy}}{x + 2\sqrt{xy}}\right)$$

2) **If cos (xy) = x + y then find $\frac{dy}{dx}$**

Solution : $\cos(xy) = x + y$

Differentiating the following w.r.t.x

$$-\sin(xy)\frac{d}{dx}xy = 1 + \frac{dy}{dx}$$

$$-\sin(xy)\left(\frac{dy}{dx} + y\right) = 1 + \frac{dy}{dx}$$

$$-x\sin(xy)\frac{dy}{dx} - y\sin(xy) = 1 + \frac{dy}{dx}$$

$$-x\sin(xy)\frac{dy}{dx} - \frac{dy}{dx} = 1 + y\sin(xy)$$

$$-(x\sin(xy) - 1)\frac{dy}{dx} = 1 + y\sin(xy)$$

$$\frac{dy}{dx} = \frac{1 + y\sin(xy)}{-(x\sin(xy) - 1)} = -\frac{1 + y\sin(xy)}{(x\sin(xy) - 1)}$$

3) **If $x^m y^n = (x + y)^{m+n}$, then find $\frac{dy}{dx}$**

Solution : $x^n y^m = (x + y)^{m+n}$

taking log on both sides, we get

$$\log x^m y^n = \log(x + y)^{m+n}$$

$$\log x^m + \log y^n = (m + n)\log(x + y)$$

$$m\log x + n\log y = (m + n)\log(x + y)$$

Differentiating the following w.r.t.x

$$m\frac{1}{x} + n\frac{1}{y}\frac{dy}{dx} = (m + n)\left(\frac{1}{(x + y)}1 + \frac{dy}{dx}\right)$$

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m + n}{(x + y)} + \frac{m + n}{(x + y)}\frac{dy}{dx}$$

$$\left(\frac{n}{y} - m + n\right)\frac{dy}{dx} = \frac{m + n}{(x + y)} - \frac{m}{x}$$

$$\left(\frac{nx + ny - my - ny}{y(x + y)}\right)\frac{dy}{dx} = \frac{mx + nx - mx - my}{(x + y)x}$$

$$\left(\frac{nx - my}{y\,(x + y)}\right) \frac{dy}{dx} = \frac{nx + - my}{(x + y)\,x}$$

$$\frac{dy}{dx} = \left(\frac{nx - my}{(x + y)\,x}\right)\left(\frac{y\,(x + y)}{nx - my}\right) = \frac{y}{x}$$

4) If $x^y = e^{x-y}$, then show that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

Solution : $x^y = e^{x-y}$

taking log on both sides, we get

$\log x^y = \log e^{x-y}$

$y \log x = (x - y) \log e = x - y$

$y \log x + y = x$

$$y = \frac{x}{(\log x + 1)}$$

Differentiating the following w.r.t.x

$$\frac{dy}{dx} = \frac{(\log x + 1)\,(1) - x\,\frac{1}{x}}{(1 + \log x)^2} = \frac{(\log x + 1) - 1}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}$$

5) If $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \tan^{-1} a$, then show that $\frac{dy}{dx} = \frac{y}{x}$

Solution : $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \tan^{-1} a$

$$\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \cos\,(\tan^{-1} a) = k \text{ (constant)}$$

$x^2 - y^2 = x^2k + y^2k \;\therefore\; x^2 - x^2k = y^2 + y^2k$

$x^2\,(1 - k) = y^2\,(1 + k)$

$$\frac{x^2}{y^2} = \frac{(1 + k)}{(1 - k)} \qquad \frac{x}{y} = \sqrt{\frac{(1 + k)}{(1 - k)}}$$ Differentiating the following w.r.t.x

$$\frac{y\,(1) - \frac{dy}{dx}}{y^2} = 0 \qquad \therefore\; y - x\,\frac{dy}{dx} = 0 \;\;\therefore\; \frac{dy}{dx} = \frac{-y}{-x} = \frac{y}{x}$$

6) **If $\log_{10}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = 2$, then show that $\frac{dy}{dx} = \frac{99\,x^2}{101\,y^2}$**

Solution : $\log_{10}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = 2 \quad \therefore \quad \frac{\log\left(\frac{x^2 - y^2}{x^2 + y^2}\right)}{\log 10} = 2$

$$\log\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = 2 \log 10$$

$$\log\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \log 100$$

$$\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = 100$$

$$x^2 - y^2 = 100\,x^2 + 100\,y^2$$

$$-99\,x^2 = 101\,y^2$$

Differentiating the following w.r.t.x

$$-99\,(2x) = 101\,(2y)\,\frac{dy}{dx}$$

$$\frac{-99\,(2x)}{101\,(2y)}\,\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = -\,\frac{-99\,x^2}{101\,y^2}$$

7) **$y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \cdots \infty}}}$ then find $\frac{dy}{dx}$**

Solution : $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \cdots \infty}}}$

Squaring on both sides, $y^2 = \log x + \sqrt{\log x + \sqrt{\log x + \cdots \infty}}$

$y^2 = \log x + y$ Differentiating the following w.r.t.x

$$2y\,\frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$$

$$\therefore (2y - 1)\,\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x\,(2y - 1)}$$

Exercise 1.4

Ex. 1) Find $\frac{dy}{dx}$ if (i) $x^5 + xy^3 + x^2 y + y^4 = 4$

(ii) $y^3 + \cos(xy) = x^2$ (iii) $x^2 + e^{xy} = y^2 + \log(x + y)$

Ex. 2) If $\log(x + y) = \log(xy) + P$, where P is a constant then prove that $\frac{dy}{dx} = -\frac{y^2}{x^2}$

Ex. 3) $e^x + e^y = e^{x+y}$ = them show that $\frac{dy}{dx} - e^{y-x}$

Ex. 4) If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \cdots \infty}}}$ then show that $\frac{dy}{dx} = \frac{\sin x}{1 - 2y}$

Ex. 5) If $e^y = y^x$ then show that $\frac{dy}{dx} = \frac{(\log x)^2}{\log y -}$

Derivative of Parametric Functions

Theorem : If $x = f(t)$ and $y = f(t)$ are two differentiable functions of parameter t, such that y is defined as function of x then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ where $\frac{dx}{dt} \neq 0$

Proof : Let δx and δy be a small increments in x and y corresponding to change δt in t.

As $\delta t \to 0$, $\delta x \to 0$, $\delta y \to 0$

x is differential function of t then $\frac{dx}{dt} = \lim_{\delta t \to 0} \frac{\delta x}{\delta t}$

y is differential function of t then $\frac{dy}{dt} = \lim \frac{\delta y}{\delta t}$

Now, $\frac{\delta y}{\delta x} = \frac{\delta y/\delta t}{\delta x/\delta t}$

Taking limit as $\delta t \to 0$

$$\therefore \lim_{\delta t \to 0} \frac{\delta y}{\delta x} = \lim_{\delta t \to 0} \frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}} = \frac{\lim_{\delta t \to 0} \frac{\delta y}{\delta t}}{\lim_{\delta t \to 0} \frac{\delta x}{\delta t}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\therefore \lim_{\delta t \to 0} \frac{\delta y}{\delta x} = \frac{dy/dt}{dx/dt} \because \delta t \to 0, \ \delta x \to 0$$

Here L.H.S. is exists so R.H.S. is exists $\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ whrer $\frac{dx}{dt} \neq 0$

1) **If $x = at^2$ and $x = 2at$, them find $\frac{dy}{dx}$**

Solution : $x = at^2$ and $x = 2at$, Differentiating the following w.r.t.x

$$\frac{dx}{dt} = 2at \text{ and } \frac{dy}{dt} = 2a$$

$$\therefore \quad \frac{dx}{dt} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

2) **If $x = a\ (\theta - \sin\theta)$ and $y = a\ (1 - \cos\theta)$, them find $\frac{dy}{dx}$**

Solution : $x = a\ (\theta - \sin\theta)$ and $y = a\ (1 - \cos\theta)$

Differentiating the following w.r.t.θ

$$\frac{dx}{d\theta} = a\ (1 - \cos\theta) \text{ and } \frac{dy}{d\theta} = a \sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\ (1 - \cos\theta)}{a \sin\theta} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = \cot\left(\frac{\theta}{2}\right)$$

3) **If $x = a \cot\theta$ and $y = b$ cosec θ, them find $\frac{dy}{dx}$**

Solution : $x = a \cot\theta$ and $y = a$ cosec θ

Differentiating the following w.r.t.θ

$$\frac{dx}{d\theta} = a\ (-\text{ cosec }\theta) \text{ and } \frac{dy}{d\theta} = b\ (-\text{ cosec }\theta \cot\theta)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\ b \text{ cosce } \theta \cot}{-\ a \text{ cosce}^2\ \theta} = \frac{b \cot\theta}{a \text{ cosce}^2\ \theta} = \cot \frac{b}{a} \cos\theta$$

4) **If $x = \sin\sqrt{t}$ and $y = e^{\sqrt{t}}$, them find $\frac{dy}{dx}$**

Solution : $x = \sin\sqrt{t}$ and $y = e\sqrt{t}$

Differentiating the following w.r.t.t

$$\frac{dx}{dt} = \cos\theta\sqrt{t}\ \frac{d}{dt}\sqrt{t} = \cos\theta\sqrt{t}\ \frac{1}{2\sqrt{t}} = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

$$\text{and } \frac{dy}{at} = e^{\sqrt{t}}\ \frac{d}{dt}\sqrt{t} = e^{\sqrt{t}}\ \frac{1}{2\sqrt{t}} = \frac{e^{\sqrt{t}}}{2\sqrt{t}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{\sqrt{t}}/2\sqrt{t}}{\cos\sqrt{t}/2\sqrt{t}} = \frac{e^{\sqrt{t}}}{\cos\sqrt{t}}$$

5) If $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$, them find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$

Solution : $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$

Differentiating the following w.r.t. θ

$\frac{dx}{d\theta} = a\, 3\cos^2 \theta \quad \frac{d}{d\theta} \cos\theta = -\,3a \cos^2 \theta \sin\theta$

$\frac{dy}{d\theta} = 3a \sin^2 \theta \quad \frac{d}{d\theta} \sin\theta = -\,3a \sin^2 \theta \cos\theta$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a \sin^2 \theta \cos\theta}{-\,3a \cos^2 \theta \sin\theta} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$$

$\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$ $\quad \therefore \left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{3}} = -\tan\theta\, \frac{\pi}{3} = -\sqrt{3}$

6) If $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$, them show that $\frac{dy}{dx} = -\frac{y}{x}^{\frac{1}{3}}$

Solution : $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$

Differentiating the following w.r.t. θ

$\frac{dx}{d\theta} = 3a \cos^2 \theta \quad \frac{d}{d\theta} \cos\theta = -\,3a \cos^2 \theta \sin\theta$

$\frac{dy}{d\theta} = 3a \sin^2 \theta \quad \frac{d}{d\theta} \sin\theta = 3a \sin^2 \theta \cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a \sin^2 \theta \cos\theta}{-\,3a \cos^2 \theta \sin\theta} = \frac{\sin\theta}{-\cos\theta}$$

Given that, $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$

$\frac{x}{a} = \cos^3 \theta$ and $\frac{y}{a} = \sin^3 \theta$

$\left(\frac{x}{a}\right)^{\frac{1}{3}} = \cos^3 \theta$ and $\left(\frac{y}{a}\right)^{\frac{1}{3}} = \sin\theta$

$$\frac{dy}{dx} = \frac{(y/a)^{\frac{1}{3}}}{-\,(x/a)^{\frac{1}{3}}} = \left(\frac{(y/a)}{-\,(x/a)}\right)^{\frac{1}{3}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

7) If $x = \log(1 + t^2)$ and $y = t - \tan^{-1} t$ them show that $\frac{dy}{dx} = \frac{\sqrt{e^x - 1}}{2}$

Solution : $x = \log(1 + t^2)$ and $y = t - \tan^{-1} t$

Differentiating the following w.r.t. t

$$\frac{dx}{dt} = \frac{1}{(1 + t^2)} \frac{d}{dt}(1 + t^2) = \frac{2}{(1 + t^2)}$$

$$\frac{dy}{dt} = 1 - \frac{1}{(1 + t^2)} = \frac{1 + t^2 \, 1 - t^2}{(1 + t^2)} = \frac{t^2}{(1 + t^2)}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{t^2}{(1 + t^2)}}{\frac{2t}{(1 + t^2)}} = \frac{t}{2}$$

Given that, $x = \log(1 + t^2) \therefore e^x = e^{\log(1 + t^2)} \therefore e^x = 1 + t^2$

$e^x - 1 = t^2 \qquad \therefore t = \sqrt{e^x - 1}$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{e^x - 1}}{2}$$

8) Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$ w.r.t. $\sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$

Solution : put $u = \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$ and $v = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$

put $x = \cos\theta \therefore \theta = \cos^{-1} x$

$$\therefore u = \tan^{-1}\left(\frac{\cos}{\sqrt{1 - \cos^2\theta}}\right) \text{ and } v = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right)$$

$$u = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right) \text{ and } v = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$u = \tan^{-1}(\cos\theta)$ and $v = \sec^{-1}(\cos 2\theta)$

$$u = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \theta\right)\right) \text{ and } v = \cos^{-1}(\cos 2\theta)$$

$u = \frac{\pi}{2} - \theta$ and $v = 2\theta$

$u = \frac{\pi}{2} - \cos^{-1} x$ and $v = 2\cos^{-1}x$ Differentiating the following w.r.t. x

$$\frac{du}{dx} = 0 - \frac{-1}{\sqrt{1 - x^2}} \text{ and } \frac{dv}{dx} = 2 \frac{-1}{\sqrt{1 - x^2}}$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1 - x^2}} \text{ and } \frac{du}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

we want to find $\frac{du}{dx} = \frac{du/dx}{du/dx}$

$$= \frac{1/\sqrt{1 - x^2}}{-2/\sqrt{1 - x^2}} = -\frac{1}{2}$$

Exercise 1.5

EX. (1) Find $\frac{dy}{dx}$ if, i) $x = at^4$, $y = 2at^2$ ii) $x = t - \sqrt{t}$, $y = t + \sqrt{t}$

iii) $x = \cos(\log t)$, $y = \log(\cos t)$ iv) $x = \sqrt{1 - x^2}$, $y = \sin^{-1} t$

EX. (2) Find $\frac{dy}{dx}$ if, i) $x = \sin^2\theta$, $y = \tan^3\theta$, at $\theta = \frac{\pi}{3}$ ii) $x = t + \frac{1}{t}$, $y = \frac{1}{t^2}$, at $t = \frac{1}{2}$

iii) $x = 3\cos t - 2\cos^3 t$, $y = 3\sin t - 2\sin^3 t$, at $t = \frac{\pi}{6}$

EX. (3) If $x = a\sqrt{\sec\theta - \tan\theta}$, $y = a\sqrt{\sec\theta + \tan\theta}$, then show that $\frac{dy}{dx} = -\frac{y}{x}$

EX. (4) If $x = e^{\sin 3t}$, $y = e^{\cos 3t}$, then show that $\frac{dy}{dx} = -\frac{y\log x}{x\log y}$

EX. (5) If $x = a\left(t - \frac{1}{t}\right)$ and $y = b\left(t + \frac{1}{t}\right)$ then show that $\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$

Higher order derivatives : If $= f(x)$ is differentiable function of x, then Differentiating the following w.r.t. x

$\frac{dy}{dx} = f'(x)$ is called the first order derivative.

Again Differentiating the following w.r.t. x

$\frac{d^2 y}{dx^2} = f''(x)$ is called the second order derivative.

Again Differentiating the following w.r.t. x

$\frac{d^3 y}{dx^3} = f'''(x)$ is called the third order derivative.

In general the n^th order derivative is $\frac{d^n y}{dx^n} = f^n(x)$

1) If $y = x^3 + 8x^2 - 3x + 10$, then find $\frac{d^2 y}{dx^2}$

Solution : $y = x^3 + 8x^2 - 3x + 10$

Differentiating the following w.r.t. x

$$\frac{dy}{dx} = 3x^2 + 16x - 3$$

Again Differentiating the following w.r.t. x

$$\frac{d^2 y}{dx^2} = 6x + 16$$

2) If $y = \sin(\log x)$, then find y_2

Solution : $y = \sin(\log x)$

Differentiating the following w.r.t. x

$$\frac{dy}{dx} = \frac{\cos(\log x)}{x}$$

Again Differentiating the following w.r.t. x

$$\frac{d^2 y}{dx^2} = \frac{x \frac{d}{dx} \cos(\log x) - \cos(\log x) \frac{d}{dx} x}{x^2} = \frac{x \frac{-\sin(\log x)}{x} - \cos(\log x)}{x^2}$$

$$\therefore y^2 = -\frac{\sin(\log x) + \cos(\log x)}{x^2}$$

3) If $x = at^2$ and $y = 2at$, then find $\frac{d^2 y}{dx^2}$

Solution : $x = at^2$ and $y = 2at$

Differentiating the following w.r.t. x

$$\frac{d}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2a} = \frac{1}{t}$$

Differentiating the following w.r.t. x

$$\frac{d}{dx} \frac{dy}{dx} = \frac{d}{dx} \frac{1}{t}$$

$$\frac{d^2 y}{dx^2} = \frac{-1}{2at^3}$$

4) If $y = \cos(m \cos^{-1}x)$, then show that $(1 - x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0$

Solution : $y = \cos(m \cos^{-1}x)$

Differentiating the following w.r.t. x

$$\frac{dy}{dx} = -\sin(m \cos^{-1}x) \frac{d}{dx}(m \cos^{-1}x) = -\sin(m \cos^{-1}x)\left(\frac{-m}{\sqrt{1-x^2}}\right)$$

$$\sqrt{1-x^2}\frac{dy}{dx} = m \sin(m \cos^{-1} x)$$

Again Differentiating the following w.r.t. x

$$\sqrt{1-x^2}\frac{d^2 y}{dx^2} + \frac{dy}{dx}\frac{1}{2\sqrt{1-x^2}}(-2x) = m \cos(m \cos^{-1} x)\left(\frac{-m}{\sqrt{1-x^2}}\right)$$

$$\therefore \sqrt{1-x^2}\frac{d^2 y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = -m^2 \cos(m \cos^{-1} x)\frac{1}{\sqrt{1-x^2}}$$

$$(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} = -m^2 \cos(m \cos^{-1} x) = -m^2 y$$

$$(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0$$

5) If $y = (x \sqrt{x^2 - 1})^m$, then show that $(x^2 - 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = m^2 y$

Solution : $y = (x \sqrt{x^2 - 1})^m$

Differentiating the following w.r.t. x

$$\frac{dy}{dx} = m\,(x + \sqrt{x^2 - 1})^{m-1}\,\frac{d}{dx}(x\sqrt{x^2 - 1})$$

$$= m\,(x + \sqrt{x^2 - 1})^{m-1}\left(1 + \frac{1}{2\sqrt{x^2 + 1}}\,2x\right)$$

$$= m\,(x + \sqrt{x^2 - 1})^{m-1}\left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right)$$

$$\frac{dy}{dx} = \frac{m\,(x + \sqrt{x^2 - 1})^{\mathrm{m}}}{2\sqrt{x^2 + 1}}$$

$$\sqrt{x^2 + 1}\,\frac{dy}{dx} = m\,(x + \sqrt{x^2 - 1})^{\mathrm{m}}$$

Again Differentiating the following w.r.t. x

$$\sqrt{x^2 + 1}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\,\frac{1}{2\sqrt{x^2 + 1}}\,2x = m\,m\,(x + \sqrt{x^2 - 1})^{m-1}\,\frac{d}{dx}(x + \sqrt{x^2 - 1})$$

$$\therefore\ \sqrt{x^2 + 1}\,\frac{d^2y}{dx^2} + \frac{x}{\sqrt{x^2 + 1}}\,\frac{d}{dx} = m^2\,(x + \sqrt{x^2 - 1})^{m-1}\left(1 + \frac{1}{2\sqrt{x^2 + 1}}\,2x\right)$$

$$\therefore\ \sqrt{x^2 + 1}\,\frac{d^2y}{dx^2} + \frac{x}{\sqrt{x^2 + 1}}\,\frac{d}{dx} = m^2\,(x + \sqrt{x^2 - 1})^{m-1}\left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right)$$

$$\therefore\ \sqrt{x^2 + 1}\,\frac{d^2y}{dx^2} + \frac{x}{\sqrt{x^2 + 1}}\,\frac{d}{dx} = \frac{m\,(x + \sqrt{x^2 - 1})^{\mathrm{m}}}{2\sqrt{x^2 + 1}}$$

$$\left(\sqrt{x^2 + 1}\right)\frac{d^2y}{dx^2} + x\,\frac{dy}{dx}\ m^2\left(x + \sqrt{x^2 - 1}\right)^m$$

$$\left(\sqrt{x^2 + 1}\right)\frac{d^2y}{dx^2} + x\,\frac{dy}{dx}\ m^2\,y$$

Exercise 1.6

Ex. 1) Find the second order derivative of the following :

(i) $x^3 + 7x^2 - 2x - 9$ (ii) x^2e^x (iii) $x^2 \log x$

Ex. 2) Find $\dfrac{d^2y}{dx^2}$ if, (i) $x = \cot^{-1}\left(\dfrac{\sqrt{1 - t^2}}{t}\right)$ and $y = \mathrm{cosec}^{-1}\left(\dfrac{\sqrt{1 - t^2}}{2t}\right)$

Ex. 3) If $x = a \cos^3 \theta$, $y = b\sin^3 \theta$, find $\frac{d^2 y}{dx^2}$ at $\theta = \frac{\pi}{4}$

Ex. 4) If $x = e^{m \tan^{-1} x}$ then show that $(1 + x^2) \frac{d^2 y}{dx^2} + (2x - m) \frac{dy}{dx} = 0$

Ex. 5) If $x = \cos t$, $y = e^{mt}$ then show that $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - m^2 y = 0$

Ex. 6) If $y = x + \tan x$ then $\cos^2 x \frac{d^2 y}{dx^2} - 2y + 2x = 0$

Ex. 7) If $y = \cos (m \cos^{-1} x)$ then show that $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0.$

Ex. 8) If $ax^2 + 2hxy + by^2 = 0$ then show that $\frac{d^2 y}{dx^2} = 0.$

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2. Applications of Derivatives

I) Application of derivative in Geometry :

Equations of tangent and Normal at $P(x_1, y_1)$ respectively are given by

$$y - y_1 = m(x - x_1) \text{ where } m = \left[\frac{dy}{dx}\right]_{(x_1,\, y_1)}$$

$$y - y_1 = m(x - x_1) \text{ where } m = -\frac{1}{\left[\frac{dy}{dx}\right]_{(x_1,\, y_1)}}, \text{ if } \left[\frac{dy}{dx}\right]_{(x_1,\, y_1)} \neq 0$$

II) Rate measure : Rate measure means derivative of the function w.r.t. time t.

Also learn that if $s = f(t)$ is displacement of a particle at time t. Then velocity $v = \frac{ds}{dt}$; accelerationa $= \frac{dv}{dt} = \frac{d^2s}{dt^2}$

III) Approximations :

Approximate value of $f(x)$ at $x = a + h$ is given by $f(a + h) \doteqdot f(a) + hf'(a)$

IV) Rolle's theorem : If real-valued function f is continuous on a $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then $\exists$ at least one $c \in (a, b)$ such that $f'(c) = 0$.

Lagrange's Mean Value Theorem (LMVT) : If real-valued function f is continuous on (a, b) and differentiable on (a, b) then $\exists$ at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{(b - a)}$$

V) Increasing and decreasing functions :

A function is increasing if $f'(x) \geq 0$ and strictly increasing if $f'(x) > 0$.

A function is decreasing if $f'(x) \leq 0$ and strictly decreasing if $f'(x) < 0$.

VI) Maxima and Minima :

A function $f(x)$ has its maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$

A function $f(x)$ has its minimum value at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$

Ex. 1) Find the equations of tangent and normal to the curve

$x^3 + 2x^2y - 9xy = 0$ at point $(2, 1)$

Solution : Given $x^3 + 2x^2y - 9xy = 0$

Differentiate *w.r.t.x.*

$$3x^2 + 2\left(x^2\frac{dy}{dx} + y\frac{d(x^2)}{dx}\right) - 9\left(x\frac{dy}{dx} + y\frac{d(x)}{dx}\right) = 0$$

$$3x^2 + 2x^2 \frac{dy}{dx} + 4xy - 9x \frac{dy}{dx} - 9y = 0$$

$$(2x^2 - 9x) \frac{dy}{dx} = 9y - 4xy - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 4xy - 3x^2}{2x^2 - 9x}$$

$$\left(\frac{dy}{dx}\right)_{(2,1)} = \frac{9(1) - 4(2)(1) - 3(2)^2}{2(2)^2 - 9(2)} = \frac{-1}{10}$$

Equation of tangent at (2, 1) is given by

$$y - 1 = \frac{-1}{10}(x - 2)$$

$$10y - 10 = -11x + 22$$

$$11x + 10y - 32 = 0$$

Equation of normal at (2, 1) is given by

$$y - 1 = \frac{-1}{\left(\frac{-1}{10}\right)}(x - 2)$$

$$y - 1 = \frac{11}{10}(x - 2)$$

$$11y - 11 = 10x - 20$$

$$10x - 11y - 9 = 0$$

Ex. 2) Find points on the curve given by $y = x^3 - 6x^2 + x + 3$ where the tangents are parallel to the line $y = x + 5$.

Solution : Given the equation of the curve $y = x^3 - 6x^2 + x + 3$

Differentiate *w.r.t.x.*

$$\frac{dy}{dx} = \frac{d\,(x^3 - 6x^2 + x + 3)}{dx} = 3x^2 - 12x + 1$$

Given that the tangent is parallel to $y = x + 5$ whose slope is 1.

$\therefore$ Slope of tangent $= \frac{dy}{dx} = 1 \Rightarrow 3x^2 - 12x + 1 = 1$

$3x(x - 4) = 0 \Rightarrow x = 0, x = 4$

When $x = 0$, $y = (0)^3 - 6(0)^2 + (0) + 3 = 3$

When $x = 0$, $y = (4)^3 - 6(4)^2 + (4) + 3 = -25$

Hence the points on the curve are (0, 3) and (4, −25)

Ex. 3) A stone is dropped in to a quiet lake and waves in the form of circles are generated, radius of circular waves increases at the rate of 5cm/sec. At the instant when the radius of circular wave is 8 cm, how fast the area enclosed is increasing?

Solution : Area of the circle $A = \pi R^2$ where R = radius of circular wave

Differentiate *w.r.t.t.*

$$\frac{dA}{dt} = \pi \frac{d(R^2)}{dt} = 2\pi R \frac{dR}{dt} \quad \ldots\ldots..(1)$$

Given that $\frac{dR}{dt}$ = 5cm/sec

When R = 8cm, substituting in (1) we get

$$\frac{dA}{dt} = 2\pi\ (8)\ (5) = 80\pi$$

Area of circular wave is increasing (when R = 8cm) at the rate 80ϖ cm^2/sec.

Ex. 4) Water is being poured at the rate of 36 m^2/sec in to a cylindrical vessel of base radius 3 m. Find the rate at which water level is rising.

Solution : Volume of cylindrical vessel $V = \pi R^2 H$

where R = radius of base

H = height of vessel. Given R = 3 m,

$V = \pi\ (3)^2\ H = 9\pi H$

Differentiate *w.r.t.t.*

$$\frac{dV}{dt} = 9\pi \frac{dH}{dt} \quad \ldots\ldots..(1)$$

Given that $\frac{dV}{dt}$ = 36 m^2/sec, substituting in (1)

$$36 = 9\pi \frac{dH}{dt}$$

$$\therefore \frac{dH}{dt} = \frac{4}{\pi}$$

thus water level is rising at the rate $\frac{4}{\pi}$ m/sec

Ex. 5) A car is moving, is given by s = $4t^2 + 3t$ where s displacement in meters, t time in seconds. What should be the velocity and the acceleration of the car at time $t = 20$ second?

Solution : Given that $s = 4t^2 + 3t$

Differentiate $w.r.t.t.$

$$v = \frac{ds}{dt} = 8t + 3 \text{ .... (1)}$$

$$\text{And } a = \frac{dv}{dt} = \frac{d(8t + 3)}{dt} = 8 \text{ .... (2)}$$

Put $t = 20$ in (1) $v = 8(20) + 3 = 163$ m/s

Put $t = 20$ in (2) $a = 8$ m/s^2

Ex. 6) Find the approximate value of following.

i) $(3.98)^3$

ii) $\sin(30^\circ\ 30')$ given that $1^\circ = 0.0175^c$, $\cos(30^\circ) = 0.866$.

iii) $e^{1.005}$ given that $e = 2.7183$

iv) $\log_{10}(998)$ given that $\log_{10}e = 0.4343$

Solution : i) observe that $a + h = 3.98$

Hence consider $a = 4$, $h = -0.02$

$\therefore f(x) = x^3$

$\therefore f'(x) = 3x^2$

$f(3.98) \doteqdot f(4) + (-0.02) f'(4)$

$(3.98)^3 \doteqdot (4)^3 - 0.02\,(3(4)^2)$

$\doteqdot 64\text{-}0.96 \doteqdot 63.04$

ii) observe that $a + h = 30^\circ\ 30'$

Hence consider $a = 30^\circ$, $h = 30' = \left(\frac{30}{60}\right)^\circ = (\frac{1}{2} \times 0.0175)^c = 0.00875$

$\therefore f(x) = \sin x$

$\therefore f(x) = \text{con } x$

$f(30^\circ\ 30') \doteqdot f(30^\circ) + (0.00875) f'(30^\circ)$

$\sin(30^\circ\ 30') \doteqdot \sin 30^\circ + 0.00875 \cos 30^\circ$

$\doteqdot \frac{1}{2} + 0.00875 \times 0.866 \doteqdot 0.575775$

iii) observe that $a + h = 1.005$

Hence consider $a = 1$, $h = 0.005$

$\therefore f(x) = e^x \quad \therefore f'(x) = e^x$

$f(1.005) \doteqdot f(1) + (0.005) f'(1)$

$e^{1.005} \doteqdot (e)^1 + 0.005\ (e^1)$

$\doteqdot 2.7183 + 0.005 \times 2.7183 \doteqdot 2.731891$

iv) Observe that $a + h = 998$

Hence consider $a = 1000$, $h = -2$

$$\therefore f(x) = \log_{10} x = \frac{\log x}{\log_{10}} = \log x \times \log_{10} e$$

$$\therefore f'(x) = \frac{1}{x} \times 0.4343$$

$f(998) \doteqdot f(1000) + (-2) f'(1000)$

$$\log_{10} 998 \doteqdot \log_{10} 1000 - 2 \left(\frac{0.4343}{1000}\right)$$

$\doteqdot \log_{10} 10^3 - 0.0008686 \doteqdot 3 \times \log_{10} 10 - 0.0008686$

$\doteqdot 3 \times 1 - 0.0008686 \doteqdot 2.9991314$

Ex. 7) Check whether conditions of Rolle's theorem are satisfied by $f(x) = x^2 - 2x + 3$, $x \in [1, 4]$

Solution : Given that $f(x) = x^2 - 2x + 3$

Since $f(x)$ is polynomial. Therefore it is continuous on $[1, 4]$ and differentiable on $(1, 4)$. Now for $a = 1$; $f(1) = (1)^2 - 2(1) + 3 = 2$

for $b = 4$; $f(4) = (4)^2 - 2(4) + 3 = 11$

$\therefore f(1) \neq f(4)$

$\therefore$ conditions of Rolle's theorem are not satisfied.

Ex. 8) Verify LMVT for the function $f(x) = \sqrt{(x + 4)}$ on the interval (0, 5)

Solution : Given that function $f(x) = \sqrt{(x + 4)}$

$f(x)$ is continuous on $(0, 5)$ and differentiable on $(0, 5)$

Therefore LMVT is applicable. Note that $a = 0$, $b = 5$

Differentiate the function $f(x)$ *w.r.t.x*

$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

$f(a) = f(0) = \sqrt{0+4} = 2$

$f(b) = f(5) = \sqrt{5+4} = 3$

By LMVT, $\exists\ c \in (0, 5)$ such that

$$f'(c) = \frac{f(\mathrm{b}) - f(a)}{(b-a)}$$

$$\frac{1}{2\sqrt{c+4}} = \frac{3-2}{5-0} \Rightarrow 5 = 2\sqrt{c+4}$$

Squaring on both sides we get $25 = 4(\mathrm{c}+4)$

$$\frac{25}{4} = \mathrm{c} + 4 \Rightarrow \mathrm{c} = \frac{25}{4} - 4 = \frac{9}{4} \in (0, 5)$$

Thus LMVT is verified.

Ex. 9) Show that the function $f(x) = x^3 + 10x + 7$ for $x \in R$ is strictly increasing.

Solution : Given the function $f(x) = x^3 + 10x + 7$

Differentiate *w.r.t.x*. $f'(x) = 3x^2 + 10$

Here note that $3x^2 \geq 0\ \forall\ x \in R$;

$\therefore\ 3x^2 + 10 \geq 10 > 0$

$\therefore\ f'(x) > 0$

$\therefore\ f(x)$ is strictly increasing function.

Ex. 10) Test whether the function $f(x) = x^3 + 6x^2 + 12x - 5$ is increasing or decreasing $\forall\ x \in R$.

Solution : Given $f(x) = x^3 + 6x^2 + 12x - 5$

$$\begin{aligned} f'(x) &= 3x^2 + 12x + 12 \\ &= 3(x^2 + 4x + 4) \\ &= 3(x+2)^2 \end{aligned}$$

Note that $(x+2)^2 \geq 0\quad \forall\ x \in R$

$3(x+2)^2 \geq 0\quad \forall\ x \in R$

$\therefore\ f'(x) \geq 0\quad \forall\ x \in R$

$\therefore\ f$ is increasing $\forall\ x \in R$

Ex. 11) Find the local maximum and local minimum value of the function.

$f(x) = x^3 - 3x^3 - 24x + 5$

Solution : Given that $f(x) = x^3 - 3x^2 - 24x + 5$

Differentiate *w.r.t.x*

$f'(x) = 3x^2 - 6x - 24$ ....(1)

Differentiate *w.r.t.x*

$f''(x) = 6x - 6$ ....(2)

Equating $f'(x) = 0$

$\therefore\ 3x^2 - 6x - 24 = 0$

$3(x^2 - 2x - 8) = 0$

$\therefore\ (x - 4)(x + 2) = 0$

$\therefore\ x = 4$ or $x = -2$

Substituting in (2), $f''(4) = 6(4) - 6 = 18 > 0$

$\therefore$ By second order derivative test, function attains its minimum value at $x = 4$. For minimum value substitute $x = 4$ in the function

$f(4) = (4)^3 - 3(4)^2 - 24(4) + 5 = -75$

Substituting in (2), $f''(-2) = 6(-2) - 6 = -18 < 0$

$\therefore$ By second order derivative test, function attains its maximum value at $x = -2$. For maximum value substitute $x = -2$ in the function.

$f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 5 = 33$

Function has maximum value 33 at $x = -2$ and minimum value -75 at $x = 4$.

Ex. 12) A wire of length 120 cm is bent in the form of a rectangle. Find its dimensions if the area of rectangle is maximum.

Solution : Let x, y be the length and breadth of rectangle in centimeters respectively.

Given perimeter of rectangle = 120 = $2(x + y)$

$\therefore\ x + y = 60 \quad \therefore y = 60 - x$

Area of rectangle $A = xy = x(60 - x) = 60x - x^2$

Differentiate *w.r.t.x* $\dfrac{dA}{dx} = 60 - 2x$

Differentiating again *w.r.t.x* $\dfrac{(d^2A)}{(dx^2)} = -2$

Now equating $\frac{dA}{dx} = 0 \Rightarrow 60 - 2x = 0$

$\therefore x = 30$

Now $\left(\frac{d^2A}{dx^2}\right)_{x = 30} = -2 < 0$

$\therefore$ By second order derivative test, area of rectangle is maximum at

$x = 30 \quad \therefore y = 60 - 30 = 30$

$\therefore$ Area of rectangle is maximum if length = 30 cm and breadth = 30 cm.

Exercise 2.1

1) A spherical soap bubble is expanding so that its radius is increasing at the rate of 0.02 cm/sec. At what rate is the surface area is increasing, when its radius is 5 cm?

2) Check the validity of the Rolle's theorem for the following functions.

   i) $f(x) = x^2 - 4x + 3, x \in [1, 3]$

   ii) $f(x) = e^{-x} \sin x, x \in [0, \pi]$

   iii) $f(x) = 2x^2 - 5x + 3, x \in [1, 3]$

   iv) $f(x) = x^{\frac{2}{3}}, x \in [-1, 1]$

3) Test whether the function $f(x) = x^3 - 6x^2 + 12x - 16, x \in R$ is increasing or decreasing.

4) Find the equations of the tangents to the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ which are parallel to the X-axis.

5) If the line $y = 4x - 5$ touches the curve $y^2 = ax^3 + b$ at the point (2, 3) find the values of a and b.

6) The surface area of a spherical balloon is increasing at the rate of 2 cm^2/sec.

   At what rate the volume of the balloon is increasing when radius of the balloon is 6 cm?

7) A man of height 2 meters walks at a uniform speed of 6 km/hr away from a lamp post of 6 meters high. Find the rate at which the length of the shadow is increasing.

8) A ladder 10 meter long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall at the rate of 1.2 meters/sec. Find how fast the top of the ladder is sliding down the wall when the bottom is 6 meters away from the wall.

9) Find the approximate value of

i) $\sqrt[3]{28}$

ii) sin (61°) given that $1^\circ = 0.0174^c$, $\sqrt{3} = 1.732$

iii) $\tan^{-1}(1.001)$

iv) $3^{2.01}$ given that $\log 3 = 1.0986$

v) $\log_e 101$ given that $\log_e 10 = 2.3026$

10) Verify Rolle's theorem for the function $f(x) = x^2 - 5x + 9$, $x \in [1, 4]$

11) Verify LMVT for the function $f(x) = 2x - x^2$, $x \in [0, 1]$

12) Find the values of x for which $f(x) = 2x^3 - 15x^2 - 144x - 7$ is strictly increasing.

13) Divide the number 30 in to two parts such that their product is maximum.

14) A ball is thrown in the air. Its height at any time t is given by $h = 3 + 14t - 5t^2$. Find the maximum height it can reach.

15) The profit function $P(x)$ of a firm, selling x items per day is given by $P(x) = (150 - x)$ x - 1625. Find the number of items the firm should manufacture to get maximum profit.

3. Indefinite Integration

Definition : If $f(x)$ and $g(x)$ are two functions such that $g'(x) = f(x)$ then we write
$\int f(x)\, dx = g(x) + c$.

For example : $\frac{d(x^3)}{dx} = 3x^2$ therefore $\int 3x^2\, dx = x^3 + c$

1) $\frac{d}{dx} = \left(\frac{x^{(n+1)}}{n+1}\right) = x^n,\ n \neq -1 \quad \therefore \int x^n dx = \frac{x^{(n+1)}}{n+1} + c$

2) $\frac{d}{dx} = \left(\frac{a^x}{\log a}\right) = a^x \quad \therefore \int a^x dx = \frac{a^x}{\log a} + c$

3) $\frac{d}{dx}(e^x) = e^x \qquad \therefore \int e^x dx = e^x + c$

4) $\frac{d}{dx} \sin x = \cos x \qquad \therefore \int \cos x dx = \sin x + c$

5) $\frac{d}{dx}(-\cos x) = \sin x \quad \therefore \int \sin x dx = -\cos x + c$

6) $\frac{d}{dx} \tan x = \sec^2 x \qquad \therefore \int \sec^2 x dx = \tan x + c$

7) $\frac{d}{dx} \sec x = \sec x \tan x \quad \therefore \int \sec x \tan x\, dx = \sec x + c$

8) $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x \qquad \therefore \int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$

9) $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x \quad \therefore \int \operatorname{cosec}^2 x\, dx = -\cot x + c$

10) $\frac{d}{dx} \log x = \frac{1}{x} \quad \therefore \int \frac{1}{x}\, dx = \log x + c$

Theorem : If f and g are real valued integrable functions of x then

1) $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$
2) $\int (f(x) - g(x))\, dx = \int f(x)\, dx - \int g(x)\, dx$
3) $\int k\,(f(x))\, dx = k \int f(x)\, dx$

Theorem : If $\int (f(x)\, dx = g(x) + c$ then, $\int f(ax + b)\, dx = \frac{1}{a} g(ax + b) + c$

Evaluate the following integrals :

1) **$\int (\sin x + \cos x)\, dx$**

Solution : $I = \int \sin x + \cos x \;\; dx$

$= \int \sin x\, dx + \int \cos x\, dx$

$= -\cos x + \sin x + c$

2) **$\int (x^2 + 3^x)\, dx$**

Solution : $I = \int (x^2 + 3^x)\, dx$

$= \int x^2\, dx + \int 3^x\, dx$

$= \dfrac{x^3}{3} + \dfrac{3^x}{log^3} + c$

3) **$\int (\tan x + \cot x)^2\, dx$**

Solution : $I = \int (\tan x + \cot x)^2\, dx$

$= \int (\tan^2 x + 2 \tan x \cot x + \cot^2 x)\, dx$

$= \int (\tan^2 x + 2 + \cot^2 x)\, dx$

$= \int (\sec^2 x - 1 + 2 + \operatorname{cosec}^2 x - 1)\, dx$

$= \int (\sec^2 x + \operatorname{cosec}^2 x)\, dx$

$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$

$= \tan x - \cot x + c$

4) **$\int \dfrac{1}{\sqrt{3x+1} - \sqrt{3x-5}}\, dx$**

Solution : $I = \int \left(\dfrac{1}{\sqrt{3x+1} - \sqrt{3x-5}}\right) dx$

$= \int \left(\dfrac{1}{\sqrt{3x+1} - \sqrt{3x-5}}\right) \cdot \left(\dfrac{\sqrt{3x+1} + \sqrt{3x-5}}{\sqrt{3x+1} + \sqrt{3x-5}}\right) dx$

$= \int \left(\dfrac{\sqrt{3x+1} + \sqrt{3x-5}}{\sqrt{3x+1} + \sqrt{3x-5}}\right) dx$

$= \int \left(\dfrac{\sqrt{3x+1} + \sqrt{3x-5}}{6}\right) dx$

$= \dfrac{1}{6} \int \left[(3x+1)^{\frac{1}{2}} + (3x-5)^{\frac{1}{2}}\right] dx$

$= \dfrac{1}{6} \left\{\int (3x+1)^{\frac{1}{2}} dx + \int (3x-5)^{\frac{1}{2}}\, dx\right\}$

$$= \frac{1}{6}\left\{\frac{(3x+1)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\cdot 3} + \frac{(3x-5)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\cdot 3}\right\} + c$$

$$= \frac{1}{18}\left\{\frac{2}{3}(3x+1)^{\frac{3}{2}} + \frac{2}{3}(3x-5)^{\frac{3}{2}}\right\} + c$$

$$= \frac{1}{27}\left\{(3x+1)^{\frac{3}{2}} + (3x-5)^{\frac{3}{2}}\right\} + c$$

5) $\int \frac{x^3}{(x-1)}\, dx$

Solution : $I = \int \frac{x^3}{(x-1)}\, dx$

$$= \int \frac{(x^3) - 1 + 1}{(x-1)}\, dx$$

$$= \int \left(\frac{x^3-1}{x-1} + \frac{1}{x-1}\right) dx$$

$$= \int \left(\frac{(x-1)(x^2+x+1}{x-1} + \frac{1}{x-1}\right) dx$$

$$= \int \left(x^2 + x + 1 + \frac{1}{x-1}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + x + \log(x-1) + c$$

6) $\int \sin 5x \cos 7x\, dx$

Solution : We know that $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$$I = \frac{1}{2}\int 2\sin 5x \cos 7x\ \ dx$$

$$= \frac{1}{2}\int [\sin(5x+7x) + \sin(5x-7x)]\, dx$$

$$= \frac{1}{2}\int [\sin(12x) + \sin(-2+)]\, dx$$

$$= \frac{1}{2}\int [\sin 12x - \sin 2x]\, dx$$

$$= \frac{1}{2}\ \ [-\cos 12x \times \frac{1}{12} + \cos 2x \times \frac{1}{2} + c$$

$$I = -\frac{1}{24}\cos 12x + \frac{1}{4}\cos 2x + c$$

7) $\int \tan^{(-1)} \left(\frac{\sin 2x}{1 + \cos 2x}\right) dx$

Solution : $I = \int \cot^{-1} \left(\frac{1 + \cos 2x}{\sin 2x}\right) dx$

$$= \int \cot^{-1} \left(\frac{2\cos^2 x}{2 \sin x \cos x}\right) dx$$

$$= \int \cot^{-1} (\cot x)\ dx$$

$$= \int x\ dx = \frac{x^2}{2} + c$$

8) **If** $f'(x) = x - \frac{3}{x^2}$, $f(1) = \frac{11}{2}$ **then find** $f(x)$

Solution : $f(x) = \int f'(x)\ dx = \int x - \frac{3}{x^2}\ dx$

$$= \frac{x^2}{2} + \frac{3}{x} + c$$

Given $f(1) = \frac{11}{2}$ $\therefore$ $\frac{1^2}{2} + \frac{3}{2} + c = \frac{11}{2}$ $\therefore$ $c = 2$

$$f(x) = \frac{x^2}{2} + \frac{3}{x} + 2$$

Exercise 3.1

Evaluate the following integrals :

1) $\int \frac{e^{2x} + e^{-2x}}{e^x} dx$

2) $\int \frac{e^{4\log x} - e^{5\log x}}{x^5} dx$

3) $\int (\tan x + \cot x)^2\ dx$

4) $\int \frac{3^x - 4^x}{5^x} dx$

5) $\int \frac{\tan x}{\sec x + \tan x} dx$

6) $\int \cos^3 x\ dx$

7) $\sqrt{1 + \sin 2x\ dx}$

8) $\int \tan^{-1} \sqrt{\frac{1 - \sin}{1 - \sin x}}$

Theorem : If $x = \phi(t)$ is a differentiable function of t , then prove that

$$\int f(x)\ dx = \int f[\phi(t)]\ \phi'(t)\ dt$$

Result : 1) $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$

2) $\int f(x)^n f'(x)\ dx = \frac{f(x)^{n+1}}{n + 1} + c$

3) $\int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + c$

Evaluate the following integrals :

1) $\int \frac{\cot(\log x)}{x}\, dx$

Solution : Let $I = \int \frac{\cot(\log x)}{x}\, dx$

put $\log x = t \quad \therefore \frac{1}{x}\, dx = 1\, dt$

$I = \int \cot t \; dt$

$= \log(\sin t) + c$

$= \log(\sin \log x) + c$

2) $\int \frac{1}{x + \sqrt{x}}\, dx$

Solution : $I = \int \frac{1}{x + \sqrt{x}}\, dx = \int \frac{1}{\sqrt{x}\,(\sqrt{x} + 1)}\, dx$

put $\sqrt{x} + 1 = t \quad \therefore \frac{1}{2\sqrt{x}}\, dx = 1dt$

$\therefore \frac{1}{\sqrt{x}}\, dx = 2dt$

$I = \int \frac{1}{t}\, 2dt = 2\int \frac{1}{t}\, dt = 2\log(t) + c$

$\therefore I = 2\log(\sqrt{x} + 1) + c$

3) $\int \frac{1}{1 + e^{-x}}\, dx$

Solution : $I = \int \frac{1}{1 + e^{-x}}\, dx = \int \frac{1}{1 + \frac{1}{e^x}}\, dx$

$I = \int \frac{1}{\frac{e^x + 1}{e^x}}\, dx = \int \frac{e^x}{e^x + 1}\, dx \quad \because \frac{d}{dx}(e^x + 1) = e^x$

$I = \log(e^x + 1) + c$

4) $\int (3x + 2)\sqrt{(x + 4)}\, dx$

Solution : Put $x - 4 = t \quad \therefore x = 4 + t \quad \therefore 1dx = 1dt$

$I = \int [3(4 + t) + 2]\sqrt{t}\, dt$

$$I = \int (14 + 3t)\ t^{\frac{1}{2}} dt = \int (14t^{\frac{1}{2}}) + 3t^{\frac{3}{2}})\ dt$$

$$I = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$= \frac{28}{3}(x-4)^{\frac{3}{2}} + \frac{6}{5}(x-4)^{\frac{5}{2}} + c$$

5) $\displaystyle\int \frac{\sin (x+a)}{\cos (x-b)}\, dx$

Solution : $\displaystyle I = \int \frac{\sin (x+a)}{\cos (x-b)}\, dx = \int \frac{\sin [(x-b)+(a+b)]}{\cos (x-b)}\, dx$

$$= \int \frac{\sin (x-b) \cos (a+b) + \cos (x-b) \sin (x+b)}{\cos (x-b)} dx$$

$$= \int \left[\frac{\sin (x-b) \cos (a+b)}{\cos (x-b)} + \frac{\cos (x-b) \sin (a+b)}{\cos (x-b)}\right] dx$$

$$= \int [\cos (a+b) \tan (x-b) + \sin (a+b)]\ dx$$

$$= \cos (a+b) \log [\sec (x-b)] + x \sin (a+b) + c$$

6) $\displaystyle\int \frac{e^x + 1}{e^x - 1}\, dx$

Solution : $\displaystyle I = \int \frac{e^x - 1 + 2}{e^x - 1}\, dx = \int \frac{e^x - 1}{e^x - 1} + \frac{2}{e^x - 1}\, dx$

$$= \int 1 + \frac{2}{e^x - 1}\, dx = \int 1\ dx + \int \frac{2}{e^x (1 - e^{-x})}\, dx$$

$$= \int 1\ dx + 2 \int \frac{e^{-x}}{1 - e^{-x}}\, dx$$

Put $(1 - e^{-x}) = t \quad \therefore -(e^{-x})(-1)dx = 1dt \quad \therefore e^{-x}dx = 1dt$

$$\therefore I = \int 1\ dx + 2 \int \frac{1}{t}\, dt = x + 2 \log(t) + c$$

$$= x + 2 \log (1 - e^{-x}) + c$$

$$\therefore \int \frac{e^x + 1}{e^x - 1}\, dx = x + 2 \log (1 - e^{-x}) + c$$

Exercise 3.2

Evaluate the following integrals :

1) $\int \dfrac{\tan(\log x)}{x} dx$ 2) $\int \dfrac{1}{x\sqrt{\log x}} dx$ 3) $\int \dfrac{sec^8 x}{cosec\ x} dx$

4) $\int e^{3\log x}(x^4+1)^{-8} dx$ 5) $\int \dfrac{e^x(1+x)}{cos(x \cdot e^x)} dx$ 6) $\int \dfrac{sin(x+a)}{cos(x+b)} dx$

7) $\int \dfrac{1}{x\log x} dx$ 8) $\int \dfrac{1}{x-\sqrt{x}} dx$

Prove that : $\int \dfrac{1}{x^2+a^2} dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$

Proof : Let, $I = \int \dfrac{1}{x^2+a^2} dx$

Put $x = a\tan\theta \therefore \tan\theta = \dfrac{x}{a} \quad \therefore \theta = \tan^{-1}\left(\dfrac{x}{a}\right)$

$\therefore dx = a\sec^2\theta d\theta$

$I = \int \dfrac{1}{a^2\tan^2\theta + a^2} a\sec^2\theta d\theta$

$= \int \dfrac{ase^2\theta}{a^2(tan^2\theta+1)} d\theta = \int \dfrac{sec^2\theta}{asec^2\theta} d\theta = \dfrac{1}{a}\int d\theta$

$= \dfrac{1}{a}\theta + c = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + c$

$\therefore \int \dfrac{1}{x^2-a^2} dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + c$

Prove that :

1) $\int \dfrac{1}{x^2-a^2} dx = \dfrac{1}{2a}\log\dfrac{x-a}{x+a} + c$ 2) $\int \dfrac{1}{a^2-x^2} dx = \dfrac{1}{2a}\log\left(\dfrac{a+x}{a-x}\right) + c$

3) $\int \dfrac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\dfrac{x}{a}\right) + c$ 4) $\int \dfrac{1}{\sqrt{x^2-a^2}} dx = \log(x+\sqrt{x^2-a^2}) + c$

5) $\int \dfrac{1}{\sqrt{x^2+a^2}} dx = \log(x+\sqrt{x^2+a^2}) + c$ 6) $\int \dfrac{1}{\sqrt{x^2-a^2}} dx = \dfrac{1}{a}\sec^{-1}\left(\dfrac{x}{a}\right)$

Evaluate the following integrals :

1) $\int \frac{1}{\sqrt{3x^2 - 7}} dx$

Solution : $I = \int \frac{1}{\sqrt{3\left(x^2 - \frac{7}{3}\right)}} dx = \int \frac{1}{\sqrt{3} \times \sqrt{x^2 - \left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}} dx$

$I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{x^2 - \left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}} dx \quad \because \int \frac{1}{x^2 - a^2} dx = \log (x + \sqrt{x^2 - a^2}) + c$

$I = \frac{1}{\sqrt{3}} \log \left[x + \sqrt{x^2 - \left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}\right] + c$

$I = \frac{1}{\sqrt{3}} \log \left[x + \sqrt{x^2 - \frac{7}{3}}\right] + c$

2) $\int \frac{1}{x^2 + 8x + 12} dx$

Solution : $I = \int \frac{1}{x^2 + 8x + 16 - 4} dx$

$I = \int \frac{1}{(x + 4)^2 - 2^2} dx \quad \because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left(\frac{x - a}{x + a}\right) + c$

$I = \int \frac{1}{2\ (2)} \log \left[\frac{(x + 4) - 2}{(x + 4) + 2}\right] + c$

$\int \frac{1}{x^2 + 8x + 12} dx = \frac{1}{4} \log \left(\frac{x + 2}{x + 6}\right) + c$

3) $\int \frac{\sin 2x}{3 \sin^4 x - 4 \sin^2 x + 1} dx$

Solution : $I = \int \frac{\sin 2x}{3 \sin^4 x - 4 \sin^2 x + 1} dx$

Put $\sin^2 x = t \quad \therefore 2 \sin x \cos x\ dx = 1dt \quad \therefore \sin^2 x\ dx = dt$

$I = \int \frac{1}{3t^2 - 4t + 1} dt = \int \frac{1}{\frac{1}{3}\left(t^2 \frac{4}{3} t + \frac{1}{3}\right)} dt$

$$= \frac{1}{3} \int \frac{1}{\left(t^2 - \frac{4}{3} t + \frac{4}{9} - \frac{4}{9} + \frac{1}{3}\right)} dt = \frac{1}{3} \int \frac{1}{\left(t^2 - \frac{4}{3} t + \frac{4}{9}\right) - \frac{1}{9}} dt$$

$$= \frac{1}{3} \int \frac{1}{\left(t - \frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2} dt = \frac{1}{3} \cdot \frac{1}{2\left(\frac{1}{3}\right)} \log \frac{\left(t^2 - \frac{2}{3}\right) - \frac{1}{3}}{\left(t^2 - \frac{2}{3}\right) + \frac{1}{3}} + c$$

$$= \frac{1}{2} \log \left(\frac{3t - 3}{3t - 3}\right) + c = \frac{1}{2} \log \left(\frac{3\sin^2 x - 3}{3\sin^2 x - 1}\right) + c$$

$$\therefore \int \frac{\sin 2x}{(3\sin^4 x - 4\sin^2 x + 1)} dx = \frac{1}{2} \log \left(\frac{3\sin^2 x - 3}{3\sin^2 x - 1}\right) + c$$

4) $\int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^{x}}} dx$

Solution : $I = \int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^{x}}} dx = \int \frac{\sqrt{e^x}}{\sqrt{\frac{1}{e^x} - e^x}} = \int \frac{\sqrt{e^x}}{\sqrt{\frac{1 - (e^x)^2}{e^x}}} dx$

$$= \int \frac{\sqrt{e^x}}{\sqrt{\frac{1 - (e^n)^2}{\sqrt{e^n}}}} dx = \int \frac{\sqrt{e^x}\sqrt{e^x}}{\sqrt{1 - (e^x)^2}} dx = \int \frac{e^x}{\sqrt{1 - (e^x)^2}}$$

Put $e^x = t \quad \therefore e^x dx = dt$

$$I = \int \frac{1}{\sqrt{1 - t^2}} dt = \sin^{-1}(t) + c = \sin^{-1}(e^x) + c$$

$$\therefore \int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^{x}}} dx = \sin^{-1}(e^x) + c$$

5) $\int \frac{1}{5 - 4 \cos x} dx$

Solution : Put $\tan \frac{x}{2} = t \quad \therefore dx = \frac{2}{1 - t^2} dt$ and $\cos x = \frac{1 - t^2}{1 - t^2}$

$$I = \int \frac{1\left(\frac{2}{1 - t^2}\right)}{5 - 4\left(\frac{2}{1 - t^2}\right)} dt = \int \frac{\frac{2}{1 - t^2}}{\frac{5(1 - t^2) - 4(1 - t^2)}{1 - t^2}} dt$$

$$= \int \frac{2}{5 + 5t^2 - 4 + 4t^2}\, dt = \int \frac{2}{9t^2 + 1}\, dt$$

$$= 2 \int \frac{1}{9\left(t^2 + \frac{1}{2}\right)}\, dt = \frac{2}{9} \int \frac{1}{t^2 + \left(\frac{1}{3}\right)^2}\, dt$$

$$= \frac{2}{9}\, \frac{1}{\left(\frac{1}{3}\right)} \tan^{-1} \frac{t}{\left(\frac{1}{3}\right)} + c = \frac{2}{3} \tan^{-1} (3t) + c$$

$$= \frac{2}{3} \tan^{-1} \left(3 \tan \frac{x}{2}\right) + c$$

$$\therefore \int \frac{1}{5 - 4\ \text{co}\ x}\, dx = \frac{2}{3} \tan^{-1} \left(3 \tan \frac{x}{2}\right) + c$$

6) $\displaystyle \int \frac{1}{3 + 2\sin^2 x + 5 \cos^2 x}\, dx$

Solution : Divide Numerator and Denominator by $\cos^2 x$

$$I = \int \frac{\frac{1}{cos^2 x}}{\frac{3 + 2sin^2 x + 5cos^2 x}{cos^2 x}}\, dx = \int \frac{sec^2 x}{3sec^2 x + 2tan^2 n + 5}\, dx$$

$$= \int \frac{sec^2 x}{3(1 + tan^2 x) + 2tan^2 x + 5}\, dx = \int \frac{sec^2 x}{5tan^2 x + 8}\, dx = \frac{1}{5} \int \frac{sec^2 x}{tan^2 x + \frac{8}{5}}\, dx$$

Put $\tan x = t \quad \therefore \sec^2 x\, dx = dt$

$$I = \frac{1}{5} \int \frac{1}{t^2 + \frac{8}{5}}\, dt = \frac{1}{5} \int \frac{1}{t^2 + \left(\frac{\sqrt{8}}{\sqrt{5}}\right)^2}\, dt$$

$$= \frac{1}{5}\, \frac{1}{\frac{\sqrt{8}}{\sqrt{5}}} \tan^{-1} \left(\frac{t}{\frac{\sqrt{8}}{\sqrt{5}}}\right) + c = \frac{1}{2\sqrt{10}} \tan^{-1} \frac{\sqrt{5}\ tanx}{2\ \sqrt{2}} + c$$

$$\therefore \int \frac{1}{3 + 2sin^2 x + 5cos^2 x}\, dx = \frac{1}{2\sqrt{10}} \tan^{-1} \frac{\sqrt{5}\ tanx}{2\ \sqrt{2}} + c$$

Exercise 3.3

Evaluate the following integrals :

1) $\int \frac{1}{x^2 + 2} dx$
2) $\int \frac{1}{4x^2 - 3} dx$
3) $\int \frac{1}{\sqrt{2x^2 - 5}} dx$
4) $\int \frac{1}{x^2 - 8x + 1} dx$
5) $\int \frac{1}{\sqrt{(x-3)(x+2)}} dx$
6) $\int \sqrt{\frac{9+x}{9-x}} dx$
7) $\int \frac{1}{4 + \cos^2 x} dx$
8) $\int \frac{\sin x}{\sin 3x} dx$
9) $\int \frac{1}{3 + 2\sin^2 x + 5\cos^2 x} dx$
10) $\int \frac{1}{3 + 2 \sin x} dx$
11) $\int \frac{3 \cos x}{4 \sin^2 x + 4 \sin x - 1} dx$

Theorem : If u and v two differentiable functions of x then

$$\int u\ v\ dx = u \int v\ dx - \int \left[\frac{du}{dx} \int v dx\right] dx$$

Evaluate the following integrals :

1) $\int x e^x\, dx$

Solution : $\int x e^x\, dx = x \int e^x\, dx - \int \left(\frac{d}{dx}(x) \int e^x\, dx\right) dx$

$= xe^x - \int (1)\, e^x\, dx$

$= xe^x - \int e^x\, dx$

$= xe^x - e^x + c$

2) $\int x \sin x\, dx$

Solution : $I = \int x \sin x\, dx$

$= x \int \sin x\, dx - \int \left(\frac{d}{dx}(x) \int \sin x\, dx\right) dx$

$= x(-\cos x) - \int (1)(-\cos x)\, dx$

$= -x \cos x + \int \cos x\, dx$

$I = -x \cos x + \sin x + c$

3) $\int e^{2x} \sin 3x\, dx$

Solution : $I = e^{2x} \int \sin^{3n} dx - \int \frac{d}{dx} e^{2x} \int \sin 3x\, dxdx$

$= e^{2x}\left(-\cos 3x \frac{1}{3}\right) - \int e^{2x}\, 2 \left(-\cos 3x \frac{1}{3}\right) dx$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x \, dx$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left(e^{2x} \int \cos 3x \, dx - \int \frac{d}{dx} e^{2x} \int \cos 3x \, dx \, dx \right)$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left(e^{2x} \left(\sin 3x \, \frac{1}{3} \right) - \int e^{2x} 2 \left(\sin 3x \, \frac{1}{3} \right) dx \right)$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} \int e^{2x} \sin 3x \, dx$$

$$I = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} I$$

$$\text{I} + \frac{4}{9} \text{I} = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x + c$$

$$\therefore \frac{13}{9} \text{I} = -\frac{e^{2x}}{3} (2 \sin 3x - 3 \cos 3x) + c$$

$$\therefore \frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + c$$

1) Evaluate : $\int x \sin^{-1} x \, dx$

Solution : $\text{I} = \int x \sin^{-1} x \, dx$

$$= \int \sin^{-1} x \int x \, dx - \int \frac{d}{dx} \sin^{-1} x \int x \, dx$$

$$= \int \sin^{-1} x \frac{x^2}{2} - \int \frac{1}{\sqrt{1 - x^2}} \frac{x^2}{2} dx$$

$$= \sin^{-1} x \frac{x^2}{2} - \frac{1}{2} \int \frac{1}{\sqrt{1 - x^2}} \, dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1 - x^2 - 1}{\sqrt{1 - x^2}} \, dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \left[\frac{1 - x^2}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - x^2}} \right] dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1 - x^2} \, dx - \frac{1}{2} \int \frac{1}{\sqrt{1 - x^2}} \, dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2} \sqrt{1 - x^2} \, dx + \frac{1}{2} x \right] - \frac{1}{2} x + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \left[\frac{x}{4} \sqrt{1 - x^2} \, dx + \frac{1}{4} x \right] + c$$

$$\therefore \int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x + \frac{x}{4} \sqrt{1 - x^2} \, dx - \frac{1}{4} x + c$$

Result : $\int e^x [f(x) + f'(x)] \, dx = e^x f(x) + c$

2) **Evaluate :** $\int e^x \left[\dfrac{x+2}{(x+3)^2}\right] dx$

Solution : $I = \int e^x \left[\dfrac{x+3-1}{(x+3)^2}\right] dx$

$$= \int e^x \left[\frac{x+3}{(x+3)^2} + \frac{-1}{(x+3)^2}\right] dx$$

$$= \int e^x \left[\frac{1}{(x+3)} + \frac{-1}{(x+3)^2}\right] dx$$

$$\therefore f(x) = \frac{1}{(x+3)} \quad \therefore f'(x) = \frac{-1}{(x+3)^2}$$

$$\therefore \int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$$

$$= e^x \left(\frac{1}{x+3}\right) + c = \left(\frac{e^x}{x+3}\right) + c$$

$$\therefore \int e^x \left[\frac{x+2}{(x+3)^2}\right] dx = \frac{e^x}{x+3} + c$$

Prove that :

1) $\int \sqrt{a^2 - x^2}\, dx = \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right) + c$

2) $\int \sqrt{a^2 + x^2}\, dx = \dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2}\log(x + \sqrt{x^2 + a^2} + c)$

3) $\int \sqrt{x^2 - a^2}\, dx = \dfrac{x}{2}\sqrt{x^2 - a^2} + \dfrac{a^2}{2}\log(x + \sqrt{x^2 - a^2} + c)$

Proof : Let $I = \int \sqrt{a^2 - x^2}\; 1\, dx$

$$= \sqrt{a^2 - x^2}\int 1\, dx - \int \left[\frac{d}{dx}\sqrt{a^2 - x^2}\int 1\, dx\right] dx$$

$$= \sqrt{a^2 - x^2}\; x + \int \frac{1}{2\sqrt{a^2 - x^2}}(-2x)(x)\, dx$$

$$= \sqrt{a^2 - x^2}\; x + \int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= \sqrt{a^2 - x^2}\; x + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}}\, dx$$

$$= \sqrt{a^2 - x^2}\; x + \int \left[\frac{a^2}{\sqrt{a^2 - x^2}} - \frac{(a^2 - x^2)}{\sqrt{a^2 - x^2}}\right] dx$$

$$= x\sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx - \int \sqrt{a^2 - x^2}\, dx$$

$$I = x\sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx - I$$

$$\therefore\ I + I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore\ I = \frac{x}{2}\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore\ \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2}\quad \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$$

Exercise 3.4

Evaluate the following integrals :

1) $\int x^2 \log x\, dx$ 2) $\int x^2 \sin 3x\, dx$ 3) $\int x \tan^{-1} x\, dx$

4) $\int e^{2x} \sin 5x\, dx$ 5) $\int \sec^3 x\, dx$ 6) $\int x^x (1 + \log x)\, dx$

7) $\int e^x [\tan x + \sec^2 x]\, dx$ 8) $\int \frac{e^x (x - 1)}{x^2} dx$ 9) $\int [\sin (\log x) + \cos (\log x)]\, dx$

Partial Fractions :

1) **Evaluate :** $\int \frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)} dx$

Solution : Let $I = \int \frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)} dx$ Let $x^2 = m$

$$\therefore\ \frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)} = \frac{2m - 3}{(m - 5)(m + 4)}$$

Now, $\frac{2m - 3}{(m - 5)(m + 4)} = \frac{A}{(m - 5)} + \frac{B}{(m + 4)} = \frac{A(m + 4) + B(m - 5)}{(m - 5)(m + 4)}$

$\therefore\ 2m - 3 = A(m + 4) + B(m - 5)$

At $m = 5$, $2(5) - 3 = A(9) + B(0)$

$7 = 9A \quad \therefore\ A = \frac{7}{9}$

At $m = -4$ $2(-4) - 3 = A(0) + B(-9)$

$-11 = -9B \ \therefore\ B = \frac{11}{9}$

Thus, $\frac{2m - 3}{(m - 5)(m + 4)} = \frac{\frac{7}{9}}{(m - 5)} + \frac{\frac{11}{9}}{(m + 4)}$

$$\therefore \frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)} = \frac{\frac{7}{9}}{x^2 - 5} + \frac{\frac{11}{9}}{x^2 + 4}$$

$$\therefore \text{ I} = \int \left[\frac{\frac{7}{9}}{x^2 - 5} + \frac{\frac{11}{9}}{x^2 + 4} \right] dx$$

$$= \frac{7}{9} \int \frac{1}{x^2 - (\sqrt{5})^2} dx + \frac{11}{9} \int \frac{1}{x^2 + (2)^2} dx$$

$$= \frac{7}{9} \frac{1}{2\sqrt{5}} \log \left[\frac{x - \sqrt{5}}{x + \sqrt{5}} \right] + \frac{11}{9} \frac{1}{2} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\therefore \text{ I} = \frac{7}{18\sqrt{5}} \log \left[\frac{x - \sqrt{5}}{x + \sqrt{5}} \right] + \frac{11}{18} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\therefore \int \frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)} dx = \frac{7}{18\sqrt{5}} \log \left[\frac{x - \sqrt{5}}{x + \sqrt{5}} \right] + \frac{11}{18} \tan^{-1} \left(\frac{x}{2} \right) + c$$

2) Evaluate : $\int \frac{1}{(\sin\theta)(3 + 2\cos\theta)} d\theta$

Solution : $\text{I} = \int \frac{1}{(\sin\theta)(3 + 2\cos\theta)} d\theta$

$$= \int \frac{\sin\theta}{(1 - \cos^2\theta)(3 + 2\cos\theta)} d\theta$$

$$= \int \frac{\sin\theta}{(1 - \cos\theta)(1 + \cos\theta)(3 + 2\cos\theta)} d\theta$$

Put $\cos\theta = t \quad \therefore -\sin\theta\, d\theta = 1dt$

$\therefore \sin\theta\, d\theta = -1dt$

Consider, $\frac{-1}{(1 - t)(1 + t)(3 + 2t)} = \frac{A}{(1 - t)} + \frac{B}{(1 + t)} + \frac{C}{(3 + 2t)}$

$$\frac{-1}{(1 - t)(1 + t)(3 + 2t)} = \frac{A(1 + t)(3 + 2t) + B(1 - t)(3 + 2t)\ C(1 - t)(1 + t)}{(1 - t)(1 + t)(3 + 2t)}$$

$$\therefore -1 = A(1 + \text{t})(3 + 2\text{t}) + B(1 - \text{t})(3 + 2\text{t}) + C(1 - \text{t})(1 + \text{t})$$

At t = 1, $-1 = A(2)(5) + B(0) + C(0)$

$-1 = 10A \quad \therefore A = -\frac{1}{10}$

At t = 1, $-1 = A\ (0) + B(2)(1) + C(0)$

$-1 = -2B \quad \therefore \quad B = -\frac{1}{2}$

At $t = -\frac{3}{2}$, $-1 = A(0) + B(0) + C\left(\frac{5}{2}\right)\left(\frac{-1}{2}\right)$

$-1 = \left(\frac{-5}{4}\right) C \quad \therefore C = \left(\frac{4}{5}\right)$

Thus, $\frac{-1}{(1-t)(1+t)(3+2t)} = \frac{\left(\frac{-1}{10}\right)}{(1-t)} + \frac{\left(\frac{-1}{2}\right)}{(1+t)} + \frac{\left(\frac{4}{5}\right)}{(3+2t)}$

$$\therefore \ I = \int \left[\frac{\left(\frac{-1}{10}\right)}{(1-t)} + \frac{\left(\frac{-1}{2}\right)}{(1+t)} + \frac{\left(\frac{4}{5}\right)}{(3+2t)}\right] dt$$

$$= -\frac{1}{10} \log(1-t)\frac{1}{(-1)} - \frac{1}{2}\log(1+t) + \frac{4}{5}\log(3+2t)\frac{1}{2} + C$$

$$= -\frac{1}{10} \log(1-\cos\theta)\frac{1}{(-1)} - \frac{1}{2}\log(1+\cos\theta) + \frac{4}{5}\log(3+2\cos\theta)\frac{1}{2} + C$$

$$= \frac{1}{10}\left(\log\frac{(1-\cos\theta)(3+2\cos\theta)^4}{(1+\cos\theta)^5}\right) + C$$

Exercise 3.5

Evaluate the following integrals :

1) $\int \frac{x^2+2}{(x-1)(x+2)(x+3)}$ 2) $\int \frac{x^2}{(x-1)(3x-1)(3x-2)}$ 3) $\int \frac{x^2+x-1}{x^2+x-6}\ dx$

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4. Definite Integration

Fundamental theorem of integral calculus :

Let f be the continuous function defined on [a,b] and if $\int f(x)dx = g(x) + c$ then

$$\int_a^b f(x)dx = [g(x) + c]_a^b$$
$$= [(g(b) + c) - (g(a)+c)]$$
$$= g(b) + c - g(a) - c$$
$$= g(b) - g(a)$$

In $\int_a^b f(x)dx$, a is called lower limit and b is called as an upper limit.

There is no need of taking the constant of integration c, because it gets eliminated.

Ex.1) : $\int_2^5 (x^2 - x)\, dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_2^5$

$$= \left[\left(\frac{5^3}{3} - \frac{5^2}{2}\right) - \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\right]$$
$$= \frac{125}{3} - \frac{25}{2} - \frac{8}{3} + \frac{4}{2}$$
$$= \frac{117}{3} - \frac{21}{2}$$
$$= \frac{234 - 8}{6}$$
$$\therefore \int_2^5 (x^2 - x)\, dx = \frac{151}{3}$$

Ex.2) : Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x\, dx$

Solution : Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x\, dx$

$$= [\sin x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3} - 1}{2}$$

Ex.3) : Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{1 - \cos 4x}\, dx$

Solution : Let $I = \int_0^{\frac{\pi}{2}} \sqrt{1 - \cos 4x}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{2\sin^2 2x}\, dx$

$$= \int_{0}^{\frac{\pi}{2}} \sqrt{2\sin^2 2x}\ dx \qquad \left(\because 1 - \cos A = 2\sin^2 \frac{A}{2}\right)$$

$$= \sqrt{2} \int_{0}^{\frac{\pi}{2}} \sin 2x\ dx$$

$$= \sqrt{2} \left[\frac{\cos^2 x}{2}\right]_0^{\pi/2}$$

$$= -\frac{\sqrt{2}}{2}\left[\cos 2\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{\sqrt{2}}{2}[\cos \pi - \cos 0]$$

$$= -\frac{\sqrt{2}}{2}[-1 -1]$$

$$= \sqrt{2}$$

$$\therefore \int_{0}^{\pi/2} \sqrt{1 - \cos 4x\ dx} = \sqrt{2}$$

Ex.4) : $\int_{-1}^{1} |5x - 3|\ .dx$

Solution : Let $I = \int_{-1}^{1} |5x - 3|.dx$

$$\therefore |5x - 3| = -(5x - 3) \text{ for } (5x - 3) < 0 \text{ i.e. } x < \frac{3}{5}$$

$$= (5x - 3) \text{ for } (5x - 3) > 0 \text{ i.e. } x > \frac{3}{5}$$

$$I = \int_{-1}^{3/5} |5x - 3|.dx + \int_{3/5}^{1} |5x - 3|.dx$$

$$= \int_{-1}^{3/5} -(5x - 3).dx + \int_{3/5}^{1} (5x - 3).dx$$

$$= \left[-\left(5\frac{x^2}{2} - 3x\right)\right]_{-1}^{3/5} + \left[\left(5\frac{x^2}{2} - 3x\right)\right]_{3/5}^{1}$$

$$= \left[3x - \frac{5}{2}x^2\right]_{1}^{3/5} + \left[\frac{5}{2}x^2 - 3x\right]_{3/5}^{1}$$

$$= \left[\left(3\left(\frac{3}{5}\right) - \frac{5}{2}\left(\frac{3}{5}\right)^2\right) - \left(3(-1) - \frac{5}{2}(-1)^2\right)\right] - \left[\left(\left(\frac{5}{2}\right) - (1)^2 - 3(1)\right) - \left(\frac{5}{2}\left(\frac{3}{5}\right)^2 - 3\left(\frac{3}{5}\right)\right)\right]$$

$= \left[\left(\frac{9}{5} - \frac{9}{10}\right) - \left(-3 - \frac{5}{2}\right)\right] + \left[\left(\frac{5}{2} - 3\right) - \left(\frac{9}{10} - \frac{9}{5}\right)\right]$

$= \frac{9}{5} - \frac{9}{10} + 3 + \frac{5}{2} + \frac{5}{2} - 3 - \frac{9}{10} + \frac{9}{5}$

$= 2\left(\frac{9}{5} - \frac{9}{10} + \frac{3}{5}\right)$

$= 2\left(\frac{18-9+25}{5}\right) = \frac{34}{5}$

$\therefore \int_{-1}^{1} |5x - 3| \, .dx = \frac{34}{5}$

Properties of Definite Integration :

Property II : $\int_a^a f(x)dx = 0$

Property II : $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Property III : $\int_a^b f(x)dx = \int_a^b f(t)dt$

Property IV : $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a < c < b$ i.e. $c \in [a, b]$

Property V : Prove that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Proof : *R.H.S.* $= \int_a^b f(a + b - x)dx$

Put $a + b - x = t$ i.e. $x = a + b - t$

$\therefore -dx = dt \Rightarrow dx = -dt$

As $x \to a \Rightarrow t \to b$ and $x \to b \Rightarrow t \to a$

$\therefore$ R.H.S. $= \int_b^a f(t)(-dt)$

$= -\int_b^a f(t)\,dt$

$= -\int_b^a f(t)\,dt$ .... $\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx)$

$= \int_0^a f(t)\,dt$ .... $\left(\because \int_a^b f(x)\,dx = -\int_a^b f(x)\,dx\right)$

= L.H.S.

Thus, $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

Property VI : Prove that $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

Proof : $R.H.S. = \int_0^a f(a - x)\, dx$

put $a - x - t$ i.e. $x = a - t$

$\therefore - dx = dt \Rightarrow dx = - dt$

As x varies from 0 to a, t varies from a to 0

$\therefore R.H.S. = \int_a^0 f(t)\,(-dt)$

$= - \int_a^0 f(t)\, dt$

$= \int_a^a f(t)\, dt$ .... $\left(\because \int_a^b f(x)\, dx = - \int_b^a f(x)\, dx\right)$

$= \int_0^a f(x)\, dx$ .... $\left(\because \int_a^b f(x)\, dx = - \int_a^b f(t)\, dt\right)$

$= L.H.S. = \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

Property VII : Prove that $\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x) dx$

Proof : $R.H.S. = \int_0^a f(x)\, dx + \int_0^a f(2a - x) dx$

$R.H.S. = I_1 + I_2$ .... (i)

Consider $I_2 = \int_0^a f(2a - x) dx$

Put $2a - x = t$ i.e. $x = 2a - t$

$\therefore - dx = dt \Rightarrow dx = - dt$

As x varies from 0 to a varies from 2a to a

$\therefore I_2 = \int_{2a}^a f(t)\,(- dt)$

$\therefore I_2 = - \int_{2a}^a f(t)\, dt$

$= \int_a^{2a} f(t)\, dt$ .... $\left(\because \int_a^b f(x)\, dx = - \int_b^a f(x)\, dx\right)$

$= \int_a^{2a} f(x)\, dx$ .... $\left(\because \int_a^b f(x)\, dx = - \int_a^b f(t)\, dt\right)$

From equation (i)

$$= R.H.S. = \int_0^a f(x)\,dx + \int_0^a f(2a - x)dx$$

$$= \int_0^a f(x)\,dx + \int_a^{2a} f(x)dx$$

$$= \int_0^{2a} f(x)dx$$

$$= L.H.S. = \text{Thus, } \int_0^{2a} f(x)\,dx = \int_0^a (x)\,dx + \int_0^a \int (2a - x)\,dx$$

Property VII : Prove that $\int_{-a}^{a} f(x) = 2\int_0^a f(x)\,dx$, if $f(x)$ is even function.
$= 0$, if $f(x)$ is odd function.

Proof : $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$ .... (i)

Consider, $I \int_{-a}^{0} f(x)\,dx$

Put $x = -t \therefore\ dx = -t$

As x varies from $-a$ to 0, t varies from a to 0.

$$I = \int_a^0 f(-t)(-dt) = -\int_a^0 f(-t)\,dt$$

$$= \int_a^0 f(-t)\,dt \text{ .... } \left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$$

$$= \int_0^a f(-x)\,dx \text{ .... } \left(\int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right)$$

equation (i) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = \int_0^a [f(-x) + f(x)]\,dx$$

if $f(x)$ is odd function then $f(-x) = -f(x)$

$$\therefore \int_{-a}^{a} f(x) = 0$$

If $f(x)$ is even function then $f(-x) = f(x)$,

$$\therefore \int_{-a}^{a} f(x) = 2\int_0^a f(x)dx$$

Hence, $\int_{-a}^{a} f(x)dx = 2\int_b^a f(x)\,dx$, if $f(x)$ is even function.

$= 0$, if $f(x)$ is odd function.

Ex.5) : Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{4}} x^3.\sin^4 x \ dx$

Solution : Let I $f(x) = x^3 \cdot \sin^4 x$

$\therefore f(-x) = (-x)^3 \cdot [\sin(-x)]^4$

$\therefore f(-x) = (-x)^3 \cdot [-\sin x]^4$

$\therefore f(-x) = (-x)^3 \cdot \ \sin^4 x$

$\therefore f(-x) = -f(x)$

$\therefore f(x)$ is odd function.

$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} x^3.\sin^4 x \ dx$

Ex.6) : Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x \ dx$

Solution : Let I $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x \ dx$ .... (i)

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 \left(\frac{\pi}{6}+\frac{\pi}{3} - x\right) dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 \left(\frac{\pi}{2} - x\right) dx$

I $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x \ dx$ .... (ii)

adding (i) and (ii) we get

$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x \ dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x \ dx$

$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sin^2 x + \cos^2 x) dx$

$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 dx$

$2I = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

$\therefore I = \frac{\pi}{12}$

$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x \ dx = \frac{\pi}{12}$

Ex.7) : Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt[3]{\tan x}}\, dx$

Solution : Let I $= \int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt[3]{\tan x}}\, dx = \int_0^{\frac{\pi}{2}} \left[\frac{1}{1+\frac{\sqrt[3]{\sin x}}{\sqrt[3]{\cos x}}}\right] dx$

$$I = \int_0^{\frac{\pi}{2}} \left[1+\frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x}+\sqrt[3]{\sin x}}\right] dx \text{ .... (i)}$$

By property $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\cos(\frac{\pi}{2}-x)}}{\sqrt[3]{\cos(\frac{\pi}{2}-x)}+\sqrt[3]{\sin(\frac{\pi}{2}-x)}}\, dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\, dx \text{ .... (ii)}$$

Adding (i) and (ii)

$$I + I = \int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{co}}{\sqrt[3]{\cos x}+\sqrt[3]{\sin x}}\, dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\, dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\cos x}+\sqrt[3]{\sin x}}{\sqrt[3]{\cos x}+\sqrt[3]{\sin x}}\, dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1.dx$$

$$I = \int \frac{1}{2}[x]_0^{\frac{\pi}{2}} = \int \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{+1\sqrt[3]{\tan x}}\, dx = \frac{\pi}{4}$$

Ex.8) : Evaluate $\int_0^{\pi} x \sin^2 x dx$

Solution : Consider, $I = {}_0^{\pi} x \sin^2 x dx$ ...... (i)

$$I = \int_0^{\pi} (\pi - x)\, [\sin(\pi - x)]^2\, dx$$

$$I = \int_0^{\pi} (\pi - x) \sin^2 x dx = \int_0^{\pi} \pi \sin^2 x dx - \int_0^{\pi} x \sin^2 x dx$$

$$I = \pi \int_0^{\pi} \frac{1}{2} (1 - \cos 2x).dx - 1 \quad ....\text{by (i)}$$

$$I + I = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x)\, dx$$

$$2I = \frac{\pi}{2} [x - \sin 2x \frac{1}{2}]_0^{\pi}$$

$$I = \frac{\pi}{4} [(\pi - \frac{1}{2} \sin 2\pi) - (0 - \frac{1}{2} \sin 0)]$$

$$= \frac{\pi}{4} [\pi] = \frac{\pi 2}{4}$$

$$\therefore \int_0^{\pi} x \sin^2 x dx = \frac{\pi^2}{4}$$

Exercise 4.1

1) $\int_{-1}^{1} |x| \, dx$

2) $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$

3) $\int_0^{\frac{\pi}{4}} \sin 4x \sin 3x \, dx$

4) $\int_0^{\frac{\pi}{4}} \sqrt{+1 \sin 2x \, dx}$

5) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1 - \sin} \, dx$

6) If $\int_0^{k} \frac{1}{1 + 8x^2} \, dx = \frac{\pi}{16}$ then find k.

7) $\int_{-4}^{2} \frac{1}{x^2 + 4x + 1} \, dx$

8) $\int_0^{1} \frac{1}{\sqrt{3 + 2x - x^2}}$

9) $\int_0^{1} x.\tan^{-1} x \, dx$

10) $\int_2^{3} \frac{\cos(\log x)}{x} \, dx$

11) $\int_1^{2} \frac{1}{x^2} e^{\frac{1}{x}} \, dx$

12) $\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{(1 - x^2)^{\frac{3}{2}}} \, dx$

13) $\int_0^{\frac{\pi}{4}} \frac{\cos x}{4x - \sin^2 x} \, dx$

14) $\int_0^{\frac{\pi}{2}} \frac{1}{5 + 4 \cos x} \, dx$

15) $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3\tan^2 + 4 \tan + 1} \, dx$

16) $\int_0^{\pi} \frac{1}{3 + 2 \sin x + \cos x} \, dx$

17) $\int_{-1}^{1} \, {}^{2}_{-4} \, \frac{1}{a^2 e^x + b^2 e^{-x}}$

18) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1 + \sin x) + (2 + \sin)} \, dx$

19) $\int_2^{e} \left[\frac{1}{\text{tog} x} - \frac{1}{\log x^2}\right] dx$

20) $\int_{-3}^{3} \frac{x^3}{(9 - x^2} \, dx$

21) $\int_{-1}^{1} \frac{x^2}{1 + x^2} \, dx$

22) $\int_0^{\frac{\pi}{2}} \log (\tan x) \, dx$

23) $\int_0^{\pi} \cos^2 x \, dx$

24) $\int_0^{\frac{\pi}{4}} \log (1 + \tan x) \, dx$

25) $\int_0^{9} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}} \, dx$

26) $\int_3^{8} \frac{(11 - x)^2}{x^2 + (11 - x)^2} \, dx$

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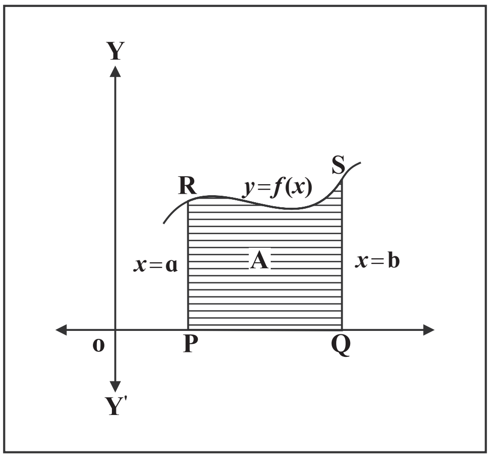
5. Applications of Definite Integration

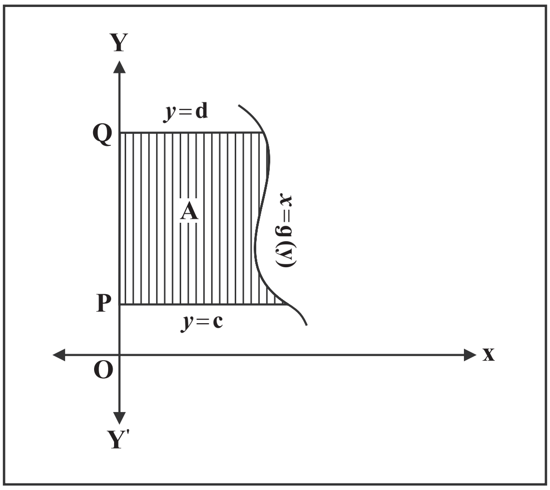
Area under a curve :

- The area A, bounded by the curve $y=f(x)$, X-axis and the lines $x=a$ and $x=b$ is given by $A = \int_a^b f(x) \, . \, dx$

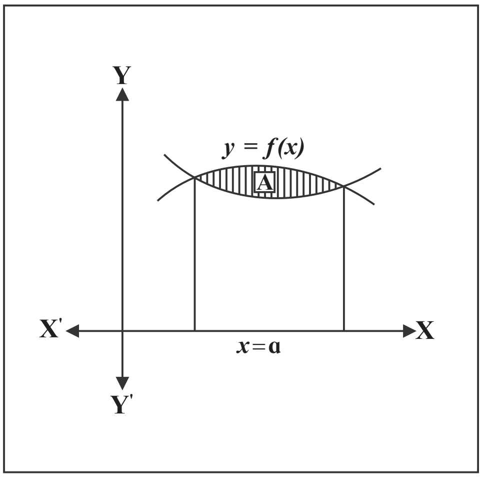


Note : If the area A lies below the X-axis, then A is negative and in this case we take $|A|$.

- The area A of the region bounded by the curve $x=g(y)$, the Y-axis, and the line $y=c$ and $y=d$ is given by $A = \int_a^b g(y) \, . \, dy$



Area between two curves :



The area bounded by the curves $y=f(x)$ and $y=g(x)$ is A $=|A_1-A_2|$

Where A_1=Area bounded by the curve $y=f(x)$, X-axis and $x=a$, $x=b$.

A2 = Area bounded by the curve $y=g(x)$, X-axis and $x=a$, $x=b$.

The point of intersection of the curves $y=f(x)$ and $y=g(x)$ can be obtained by solving their equations simultaneously.

$\therefore$ The required area $A=|\int_a^b f(x)\,dx - \int_a^b f(x)\,dx|$

Ex.1) Find the area bounded by the curve $x^2=y$, the Y axis the X axis and $x=3$.

Solution :

The required area

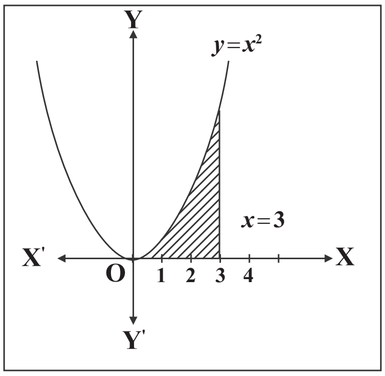
$$A = \int_{x=0}^{3} y\,dx.$$

$$= \int_0^3 x^2\,dx.$$

$$= \left[\frac{x^3}{3}\right]_0^3$$

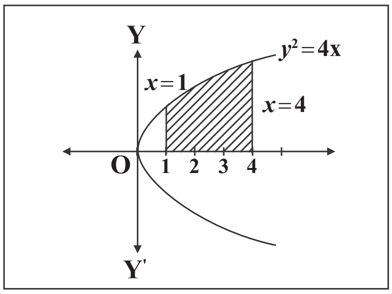
$$A = 9 - 0$$

$\therefore A = 9$ sq. unit



Ex.2) Find the area of the region bounded by the curve $y^2=4x$, the X axis and the lines $x=1$, $x=4$, $y \geq 0$

Solution :



$\therefore$ The required area $= A = \int_1^4 y\,dx$

$$= \int_1^4 2\sqrt{x}\,dx$$

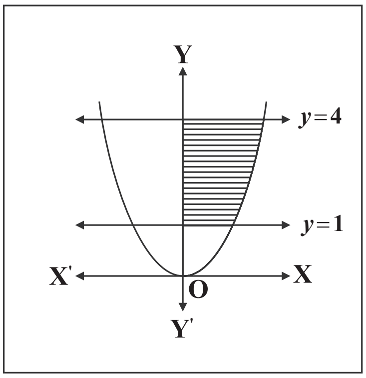
$$= 2\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$$

$$= \frac{4}{3}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$\therefore A = \frac{28}{3} \text{ sq.unit}$$

Ex.3) Find the area of the region bounded by the curves $x^2=16y$, $y=1$, $y=4$ and the Y-axis, lying in the first quadrant.

Solution :



Required area $= \int_1^4 x\,dy = \int_1^4 \sqrt{16y}\,dy$

$$= \int_1^4 4\sqrt{y}\,dy$$

$$= 4\left[\frac{2}{3}y^{\frac{3}{2}}\right]_1^4 = \frac{8}{3}[8-1] = \frac{56}{3}$$

$$\therefore \text{A} = \frac{56}{3} \text{ sq.unit}$$

Ex.4) Find the area of the region bounded by curve $y=2x$, X-axis and the lines $x=0$, $x=5$.

Solution :

Required area $= A = \int_0^5 y\,dx = \int_0^5 2x\,dx$

$$= 2\left[\frac{x^2}{2}\right]_0^5 = 2\left[\frac{25}{2} - 0\right] = 25 \text{ sq.unit}$$

Ex.5) Find the area the circle $x^2+y^2=16$

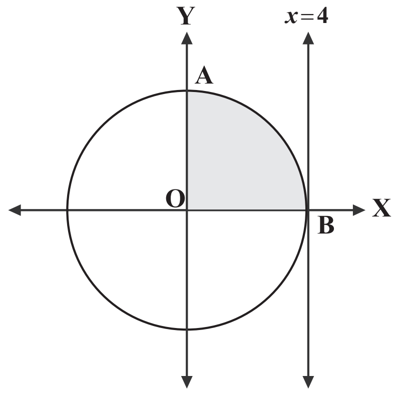
Solution :

From the equation of circle, $x^2+y^2=16$ we get

$y^2 = (16-x^2)$

$y = \sqrt{16-x^2}$

$\therefore$ By the symmetry of the circle,



$\therefore$ Required area A = 4A(region$O\ ABO$)

$= 4\int_0^4 y\,dx = 4\int_0^4 \sqrt{16-x^2}\,dx$

$= 4\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$

$= 4 = \left[8\,\frac{\pi}{2} - 0\right] = 16\pi$

$\therefore$ Area of the circle is 16π sq. unit.

Ex.6) Find the area of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

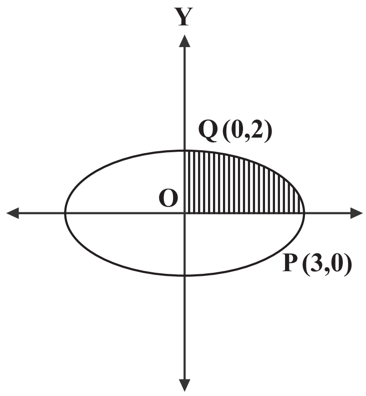
Solution :

From the equation of ellipse, $\frac{x^2}{9} + \frac{y^2}{4} = 1$ we get

$y^2 = \frac{4}{9}(9 - x^2)$

$\therefore \quad y = \frac{2}{3}\sqrt{9-x^2}$

$\therefore$ By the symmetry of the ellipse,



$\therefore$ Required area A = 4A(region$O\ PQO$)

$= 4\int_0^3 y\,dx = 4\int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx$

$= \frac{4\,(2)}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$

$= \frac{8}{3}\left[\frac{9}{2}\,\frac{\pi}{2} - 0\right] = 6\pi$

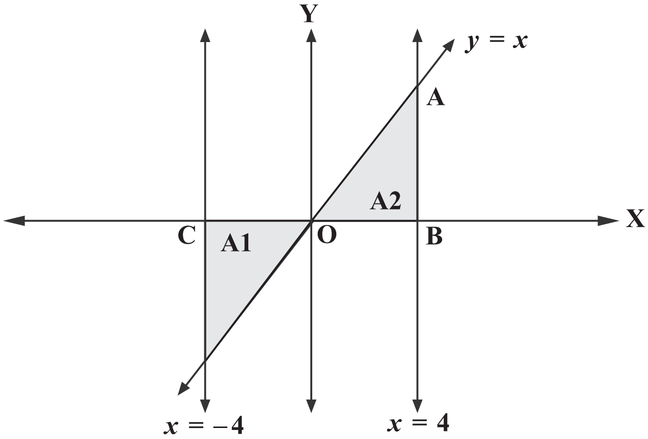
$\therefore$ Area of the ellipse is 6π sq. unit.

Ex.7) Find the area bounded by the line $y = x$, X-axis and the lines $x = -4$ and $x = 4$.

Solution :

Consider the area A, bounded by straight line $y = x$, X-axis and $x = 4$, $x = 4$.

From figure, A is divided into A_1 and A_2



The required area A1= $\int_{4}^{0} y\,dx = \left[\frac{x^2}{2}\right]_{-4}^{0} = 0 - \frac{16}{2} = -8$

But area is always positive.

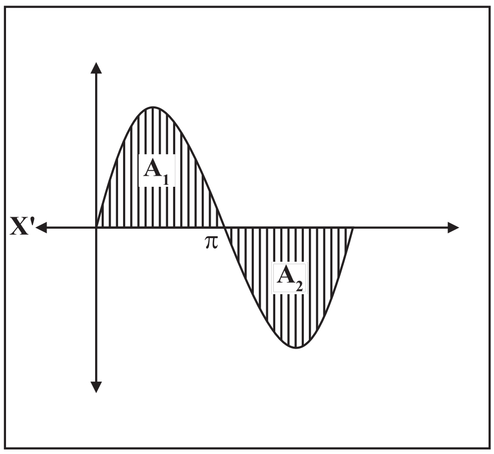
$\therefore A_1 = |-8|$ sq. unit = 8 sq. unit.

$\therefore A_2 = \int_{4}^{0} ydx = \int_{4}^{0} xdx = \left[\frac{x^2}{2}\right]_{0}^{4} = \frac{4^2}{2} = 8$ square unit.

$\therefore$ Required area $A = A_1 + A_2 = 8 + 8 = 16$ square unit.

Ex.8) Find the area enclosed between the X-axis and the curve $y = \sin x$ for values of x between 0 to 2π

Solution :



Area A_1 = area lying above the X-axis

$= \int_{0}^{\pi} \sin x.dx = [-\cos x]_{0}^{\pi}$

$= -[\cos\pi - \cos 0] = -(-1-1)$

$A_1 = 2$

Area A_2= area lying below the X-axis

$= \int_{\pi}^{2\pi} \sin xdx = [-\cos x]$

$= \quad [-\cos 2\pi - \cos\pi]$

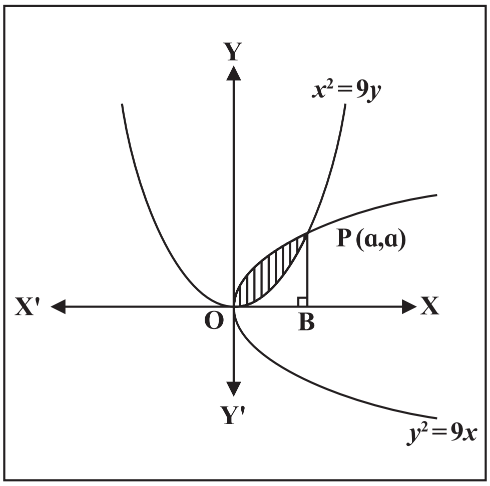
$= \quad -[-1-(-1)]$

$= \quad -2$

$\therefore$ Total area = A1 + $|A_2|$ = 2 + |(−2) = 4 sq. unit

Ex.9) Find the area of the region bounded by the curves $y^2=9x$ and $x^2=9y$.

Solution :



The equations of the curves are

$y^2 = 9x$ .... (I)

$x^2 = 9y$ .... (II)

Squaring equation (II)

$x^4 = 81y^2$

$x^4 = 81(9x)$ .... by (I)

$x^4 = 729x$

$\therefore x=0 \quad ; x=9$

From equation (II), $y = 0$ or $y = 9$

$\therefore$ The points of intersection of the curves are (0, 0), (9, 9).

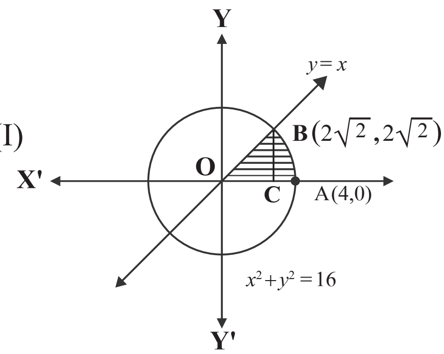
$$\therefore \text{ required area} = \int_0^9 \sqrt{9\,x\text{dx}} - \int_0^9 \frac{x^2}{9}\,dx = \left[3.\frac{2}{3}.x^{\frac{3}{2}}\right]_0^9 - \left[\frac{1}{9}\cdot\frac{x^3}{3}\right]_0^9$$

$$= 2\left[9^{\frac{3}{2}} - 0\right] - \frac{1}{27}[9^3 - 0]$$

$$= 2.9^{\frac{3}{2}} - 27 = 54 - 27 \qquad \therefore A = 27 \text{ sq. unit}$$

Ex.10) Find the area of sector bounded by the circle $x^2+y^2=16$ and the line $y = x$ in the first quadrant.

Solution :



Required area A = $A(\Delta OCB)$ + A(region ABC)

To find, the point of intersection of $x^2+y^2=16$ ..... (I)

and the line $y = x$ .... (II)

Substitute (II) in (I)

$x^2 + x^2 = 16$

$2x^2 = 16$

$x^2 = 8,$

$x = \pm\ 2\sqrt{2}\ ,\ \ y = \pm\ 2\sqrt{2}$

The point of intersection is B $(\ 2\sqrt{2}\ ,\ \ 2\sqrt{2}\ )$

$$A = \int_0^{2\sqrt{2}} x\ dx + \int_{2\sqrt{2}}^{0} \sqrt{16 - x^2}\ dx$$

$$= \frac{1}{2}\ [x^2]_0^{2\sqrt{2}} + \left[\frac{x}{2}\ \sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]\ 2\sqrt{2}$$

$$= \frac{1}{2}\ .\ (\ 2\sqrt{2}\ )^2 + \left[8\ \sin^{-1}\ 1 - \left\{\frac{2\sqrt{2}}{4}\ \sqrt{8} + 8\ \sin^{-1}\frac{1}{2}\right\}\right]$$

$$= 4 + 8.\frac{\pi}{2} - 4 - 8.\frac{\pi}{4}$$

$\therefore\ A = 2\pi$ sq. unit.

Exercise 5.1

1) Find the area of the region bounded by the following curves, X-axis and the given lines.

i) $x = 2y,\ x = 0,\ x = 4$

ii) $x = 0,\ x = 5,\ y = 0,\ y = 4$

iii) $y = \sin x,\ x = 0,\ x = \frac{\pi}{2}$

iv) $y^2 = 16x$ and $x = 0,\ x = 4$

v) $y = -\ x^2,\ x = 1$ and $x = 4$

vi) $2y + x = 8,\ x = 2$ and $x = 4$

2) Find the area bounded by the line $y = 2x$, X-axis and the lines $x = -\ 2$ and $x = 4$.

3) Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

4) Find the area of the ellipse $= \frac{x^2}{25} + \frac{y^2}{16} = 1$

5) Find the area the circle $x^2 + y^2 = 25$

6) Find the area of the region included between $y^2 = 2x$ and the line $y = 2x$.

◈◈◈

6. Differential Equations

1) An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a **differential equation**.
2) The order of highest derivative occurring in the differential equation is called **order** of the differential equation.
3) The power of the highest ordered derivative when all the derivatives are made free from negative and / or fractional indices if any is called **degree** of the differential equation

   Note :

   - To find the degree of the differential equation, we need to have a positive integer as the index of each derivative.
   - The Order and degree (if defined) of a differential equation are always positive integers.
   - The order of a differential equation representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.
   - We shall prefer to use the following notations for derivatives :
     $\frac{dy}{dx} = y'$, $\frac{d^2y}{dx^2} = y''$, $\frac{d^3y}{dx^2} = y'''$ and so on.

4) A function which satisfies the given differential equation is called its **solution**. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called **particular solution**.
5) To form a differential equation from a given function we **differentiate** the function successively as many times as the **number of arbitrary constants** in the given function and then eliminate the arbitrary constants.
6) **Variable separable method** is used to solve such an equation in which variables can be separated completely i.e. terms containing y should remain with dy and terms containing x should remain with dx.
7) If the **homogeneous differential equation** is in the form $\frac{dy}{dx} = F(x, y)$ where, $F(x, y)$ is homogenous function of degree zero, then we make substitution $\frac{y}{x} = v$ i.e. $y = vx$ and we proceed further to find the general solution by writing $\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right)$.
8) If the **homogeneous differential equation** is in the form $\frac{dx}{dy} = F(x, y)$ where, $F(x, y)$ is homogenous function of degree zero, then we make substitution $\frac{x}{y} = v$ i.e. $x = vy$ and we proceed further to find the general solution by writing $\frac{dx}{dy} = F(x, y) = g\left(\frac{x}{y}\right)$.

9) The most general form of a **linear differential equation** of the first order is :

$\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only, is known as a first order linear differential equation.

Steps involved to solve first order linear differential equation :

i) Write the given differential equation in the form : $\frac{dy}{dx} + Py = Q$ where P, Q are constants or functions of x only.

ii) Find the Integrating Factor (I.F) $= e^{\int pdx}$.

iii) Write the solution of the given differential equation as $y(\text{I.F}) = \int[Q \times (\text{I.F.})]\, dx + C$

10) The most general form of a **linear differential equation** of the first order is :

$\frac{dx}{dy} + Px = Q$, where P and Q are constants or functions of y only, is known as a first order linear differential equation.

Steps involved to solve first order linear differential equation :

i) Write the given differential equation in the form : $\frac{dx}{dy} + Px = Q$
where P, Q are constants or functions of y only.

ii) Find the Integrating Factor (I.F) $= e^{\int pdy}$

iii) Write the solution of the given differential equation as $x(\text{I.F}) = \int[Q \times (\text{I.F.})]\, dy + C$

11) **Application of differential Equations :**

Differential equations can be used to describe mathematical models such as population expansion or radioactive decay etc. Some applications of differential equation are:

a) Population Growth
b) Growth of Bacteria
c) Radio Active Decay
d) Half Life Period
e) Newton's Law of Cooling
f) Surface Area

Ex. 1) Find the order and degree, if defined, of the following differential equations :

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = \sqrt{1 + \frac{d^3y}{dx^3}}$$

Solution : This equation expressed as $\left(\frac{d^2y}{dx^2} + \frac{dy}{dx} + x\right)^2 = 1 + \frac{d^3y}{dx^3}$

The highest order derivative present in the given differential equation is $\frac{d^3y}{dx^3}$, so its **order is three**. It is a polynomial equation in $\frac{d^3y}{dx^3}$, $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and the highest power raised to $\frac{d^3y}{dx^3}$ is one, so its **degree is one**.

Ex. 2) Obtain the differential equation by eliminating arbitrary constants form the following :

$\boldsymbol{y = c_1e^{3x} + c_2e^{2x}}$

Solution : $y = c_1e^{3x} + c_2e^{2x}$ ...(I)

Differentiate *w. r. t. x*, we get

$$\frac{dy}{dx} = 3c_1e^{3x} + 2c_2e^{2x} \quad \text{...(II)}$$

Again differentiate *w. r. t. x*, we get

$$\frac{d^2y}{dx^2} = 9c_1e^{3x} + 4c_2e^{2x} \quad \text{...(III)}$$

As equations (I), (II) and (III) in c_1e^{3x} and c_2e^{2x} are consistent

$$\therefore \begin{vmatrix} y & 1 & 1 \\ \frac{dy}{dx} & 3 & 2 \\ \frac{d^2y}{dx^2} & 9 & 4 \end{vmatrix} = 0$$

$$\therefore y(12 - 18) - 1\left(4\frac{dy}{dx} - 2\frac{d^2y}{dx^2}\right) + 1\left(9\frac{dy}{dx} - 3\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore -6y - 4\frac{dy}{dx} + 2\frac{d^2y}{dx^2} + 9\frac{dy}{dx} - 3\frac{d^2y}{dx^2} = 0$$

$$\therefore -\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$$

$$\therefore \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$ is the required differential equation.

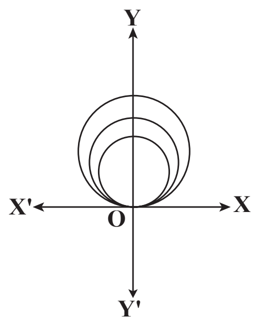
Ex. 3) From the differential equation of family of circles above the X-axis and touching the X-axis at the origin.

Solution : Let C (a, b) be the center of the circle touching X-axis at the origin (b < 0). The radius of the circle is b. The equation of the circle is

$(x - 0)^2 + (y - b)^2 = b^2$

$\therefore x^2 + y^2 - 2by + b^2 = b^2$

$\therefore x^2 + y^2 - 2by = 0$ ....(I)



Differentiate *w. r. t. x*, we get

$$2x + 2y\left(\frac{dy}{dx}\right) - 2b\left(\frac{dy}{dx}\right) = 0$$

$\therefore \ \underline{x} + (y - b)\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{x}{\dfrac{dy}{dx}} + (y - b) = 0$

$\therefore \ b = y + \dfrac{x}{\dfrac{dy}{dx}}$ ...(II)

From eEx. (I) and eEx. (II)

$\therefore \ x^2 + y^2 - 2\left[y + \dfrac{x}{\dfrac{dy}{dx}}\right] y = 0$

$\therefore \ x^2 + y^2 - 2y^2 - \dfrac{2xy}{\dfrac{dy}{dx}} = 0$

$\therefore \ x^2 - y^2 = \dfrac{2xy}{\dfrac{dy}{dx}}$

$\therefore (x^2 - y^2)\dfrac{dy}{dx} = 2xy$ is the required differential equation.

Ex. 4) Verify that : $y = \log x + c$ is a solution of the differential equation $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$.

Solution : Here $y = \log x + c$

Differentiate w. r. t. x, we get

$\dfrac{dy}{dx} = \dfrac{1}{x}$

$\therefore \ x\dfrac{dy}{dx} = 16$

Differentiate w .r .t .x ,

we get $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

Hence $y = \log x + c$ is a solution of the differential equation

$x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

Ex. 5) Find the general solution of the differential equation : $\frac{dx}{dt} = \frac{xlogx}{t}$.

Solution : $\frac{dx}{dt} = \frac{xlogx}{t}$

$\therefore \frac{dx}{xlogx} = \frac{dt}{t}$

Integrating both sides, we get

$\therefore \int \frac{dx}{xlogx} = \int \frac{dt}{t}$

$\therefore log(log\ x) = log\ (t) + \log c$

$\therefore log(log\ x) = log\ (tc)$

$\therefore log\ x = ct$

$\therefore e^{ct} = x$

Ex. 6) Find the particular solution of the differential equation $\frac{y-1}{y+1} + \frac{x-1}{x+1} \cdot \frac{dy}{dx} = 0$, when $x = y = 2$

Solution : $\frac{y-1}{y+1} + \frac{x-1}{x+1} \cdot \frac{dy}{dx} = 0$

$\therefore \frac{x+1}{x-1}dx + \frac{y+1}{y-1}\text{dy} = 0$

$\therefore \frac{(x-1)+2}{x-1}dx + \frac{(y-1)+2}{y-1}dy = 0$

$\therefore \left(1 + \frac{2}{x-1}\right)dx + \left(1 + \frac{2}{x-1}\right)dy = 0$

Integrating, we get

$\therefore \int dx + 2\int \frac{dx}{x-1} + \int dy + 2\int \frac{dy}{y-1} = 0$

$\therefore x + 2\ log\ (x-1) + y + 2\ log\ (y-1) = c$

$\therefore x + y + 2\ log\ [(x-1)(y-1)] = c$ .... (I)

When $x = 2$, $y = 2$. So eqn. (I) becomes

$\therefore 2 + 2 + 2\ log\ [(2-1)(2-1)] = c$

$\therefore 4 + 2\ log(1 \times 1) = c$

$\therefore 4 + 2\ log(1) = c$

$\therefore 4 + 2(0) = c$

$\therefore c = 4$

Put in eEx. (I), we get

$\therefore x + y + 2log\ [(x-1)(y-1)] = 4$ is the required particular solution.

Ex. 7) Reduce each of the following differential equations to the separated variable form and hence find the general solution $\frac{dy}{dx} = (4x + y + 1)^2$

Solution : $\frac{dy}{dx} = (4x + y + 1)^2$

Put $4x + y + 1 = u$

$\therefore\ 4 + \frac{dy}{dx} = \frac{du}{dx}$

$\therefore\ \frac{dy}{dx} = \frac{du}{dx} - 4$

Given differential equation becomes

$\frac{du}{dx} - 4 = u^2$

$\therefore \frac{du}{dx} = u^2 + 4$

$\therefore \frac{du}{u^2 + 4} = dx$

Integrating both sides we get

$\int \frac{du}{u^2 + 4} = \int dx$

$\therefore\ \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) = x + c_1$

$\therefore \tan^{-1}\left(\frac{u}{2}\right) = 2x + 2c_1$

$\therefore \tan^{-1}\left(\frac{4x + y + 1}{2}\right) = 2x + c$ ..... (Put $2c_1 = c$)

Ex. 8) Solve the differential equations $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$

Solution : $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$ .....(I)

It is homogeneous differential equation

Put $y = vx$ .....(II)

Differentiate w. r. t. x. we get

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ....(III)

Put (II) and (III) in equation (I) we get,

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2 x^2}}{x}$$

$$\therefore\ v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$$

$$\therefore\ x\frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\therefore\ \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

Integrating both sides we get

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

$$\therefore\ \log\left(v + \sqrt{1 + v^2}\right) = \log x + \log c$$

$$\therefore\ \log\left(v + \sqrt{1 + v^2}\right) = \log cx$$

$$\therefore\ v + \sqrt{1 + v^2} = cx$$

$$\therefore \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = cx$$

$\therefore\ y + \sqrt{x^2 + y^2} = cx^2$ is the solution.

Ex. 9) Solve the differential equation : $x \sin x\frac{dy}{dx} + (x \cos x + \sin x)y = \sin x$

Solution : $x \sin x\frac{dy}{dx} + (x \cos x + \sin x)y = \sin x$

divide by $x \sin x$ we get

$$\frac{dy}{dx} + \left(\cot x + \frac{1}{x}\right) y = \frac{1}{x} \quad \text{......(I)}$$

It is the linear differential equation of the type

$$\frac{dy}{dx} + Py = Q$$

where, $P = \cot x + \frac{1}{x}$, $Q = \frac{1}{x}$

Its solution is

$$y(I.F.) = \int Q\ .\ (I.F.)dx + c \quad \text{.....(II)}$$

where, $\quad (\text{I.F.}) = e^{\int P.dx} = e^{\int\left(\cot x + \frac{1}{x}\right) . dx}$

$(I.F.) = e^{\int \cot x .\, dx + \int \frac{1}{x} . dx}$

$(I.F.) = e^{log|\sin x| + log x}$

$(I.F.) = x \sin x$

equation (II) becomes,

$y \,.\, x \sin x = \int \frac{1}{x} . x \sin x \,.\, dx + c$

$y \,.\, x \sin x = - \cos x + c$

$\therefore\ y \,.\, x \sin x + \cos x = c$ is the general solution.

Ex. 10) The slope of the tangent to the curve at any point is equal to $y + 2x$. Find the equation of the curve passing through the origin.

Solution : Let, $P(x, y)$ be any point on the curve $y = f(x)$

The slope of the tangent at point $P(x, y)$ is $\frac{dy}{dx}$

$\therefore\ \frac{dy}{dx} = y + 2x$

$\therefore\ \frac{dy}{dx} - y = 2x$

This is linear differential equation of the type $\frac{dy}{dx} + Py = Q$

where, $P = -1,\ Q = 2x$

It's solution is, $y\ (I.F) = \int[Q \times (I.F.)]\ dx + C$ ......(I)

where, $\quad (I.F.) = e^{\int P.dy}$

$= e^{\int -1.dx}$

$(I.F.) = e^{-x}$

equation (I) becomes,

$y \,.\, e^{-x} = \int 2xe^{-x} .\, dx + c$

$y \,.\, e^{-x} = \int 2xe^{-x} .\, dx + c$ ......(II)

Consider, $\int xe^{-x} dx$

$= x\int e^{-x} .\, dx - \int \left[1 \times \frac{e^{-x}}{-1}\right] dx$

$= \frac{x \,.\, e^{-x}}{-1} + \int e^{-x} dx$

$= -x \,.\, e^{-x} + \int e^{-x}dx$

$= -x \,.\, e^{-x} - e^{-x}$

equation (II) becomes,

$y \,.\, e^{-x} = 2(-x \,.\, e^{-x} - e^{-x}) + c$

$\therefore\ y \,.\, e^{-x} = -2xe^{-x} - 2e^{-x}) + c$ ......(III)

The curve passes through the origin (0, 0)

i.e. put $x = 0$ and $y = 0$ in equation (III)

$\therefore\ 0 \,.\, e^{-0} = -2(0)e^{-0} - 2e^{-0} + c$

$\therefore 0 = -2 + c$

$\therefore\ 2 = c$ Put in (III)

$y \,.\, e^{-x} = -2xe^{-x} - 2e^{-x} + 2$

$\therefore\ y = -2x - 2 + 2e^{-x}$

$\therefore 2x + y + 2 = 2e^{-x}$ is the equation of the curve.

Ex. 11) The population of a town increasing at a rate proportional to the population at that time. If the population increases from 40 thousands to 60 thousands in 40 years, what will be the population in another 20 years? $\left(\textbf{Given } \frac{3}{2} = 1.2247\right)$**.**

Solution : Let P be the population at time t. Since rate of increase of P is a proportional to P itself, we have,

$$\frac{dp}{dt} = k \cdot P \qquad \ldots(1)$$

Where k is constant of proportionality.

Solving this differential equation, we get

$P = a \cdot e^{kt}$, where $a = e^{c} \ldots(2)$

Initially $P = 40{,}000$ when $t = 0$

$\therefore$ From equation (2), we have

$40{,}000 = a \cdot 1 \qquad \therefore a = 40{,}000$

$\therefore$ Equation (2) becomes

$P = 40{,}000 \cdot e^{kt} \ldots(3)$

Again given that $P = 60{,}000$ when $t = 40$

$\therefore$ From equation (3), $60{,}000 = 40{,}000 \cdot e^{40k}$

$$\therefore \quad e^{40k} = \frac{3}{2} \ldots(4)$$

Now we have to find P when $t = 40 + 20 = 60$ years.

$\therefore$ From equation (3), we have

$$P = 40{,}000 \cdot e^{60k} = 40{,}000\ (e^{40k})^{\frac{3}{2}}$$

$$= 40{,}000 \left(\frac{3}{2}\right)^{\frac{3}{2}} = 73{,}482$$

$\therefore$ Required population will be 73,482.

Ex. 12) Bismath has half life of 5 days. A sample originally has a mass of 800mg. Find the mass remaining after 30 days.

Solution : Let x be the mass of Bismath present at time t.

Then $\dfrac{dx}{dt} = -kx$ where $k > 0$

Solving the differential equation, we get

$x = c \cdot e^{-k}$ ......(1)

where c is the constant of proportionality.

Given that $x = 800$ when $t = 0$

Using these values in equation (1), we get

$800 = c \cdot 1 = c$

$\therefore x = 800\ e^{-kt} \ldots(2)$

Since half life is 5 days, we have $x = 400$ when $= 5$,

$\therefore$ From equation (2), we have

$400 = 800 \cdot e^{-5k}$

$$\therefore e^{-5k} = \frac{400}{800} = \frac{1}{2} \ldots.(3)$$

Now we have to determine x when $= 30$,

$\therefore$ From equation (2), we have

$x = 800\ e^{-30k} = 12.5$

$\therefore$ The mass after 30 days will be 12.5 mg.

Ex. 13) Water at 100°c cools in 10 minutes to 88°c in a room temperature of 25°c. Find the temperature of water after 20 minutes.

Solution : Let θ be the temperature of water at time t. Room temperature is given to be 25°c. Then according to Newton's law of cooling, we have $\frac{d\theta}{dt} \propto (\theta - 25)$

$\therefore \frac{d\theta}{dt} = -k(\theta - 25)$, where $k > 0$

After integrating and using initial condition, we get

$\theta = 25 + 75 \cdot e^{-kt}$ ……(1)

But given that $\theta = 88°c$ when $t = 10$

$\therefore$ From equation (1) we get

$88 = 25 + 75 \cdot e^{-10k}$

$\therefore 63 = 75 \cdot e^{-10k}$

$\therefore e^{-10k} = \frac{63}{75} = \frac{21}{25}$ ……..(2)

Now we have to find θ when $t = 20$

$\therefore$ From equation (1) we have

$\theta = 25 + 75 \cdot e^{-20k}$

$= 25 + 75 \cdot (e^{-10k})^2$

$= 25 + 75 \cdot \left(\frac{21}{25}\right)^2$ …..by (2)

$= 25 + 75 \times \frac{21}{25} \times \frac{21}{25}$

$= 25 + \frac{1323}{25}$

$= 77.92$

$\therefore$ Temperature of water after 20 minutes will be 77.92°c .

Exercise 6.1

1) Find the degree and order of differential equation.

a) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 8\frac{d^2y}{dx^2}$. b) $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

2) Obtain the differential equation by eliminating arbitrary constants form the following:

a) $y = A\cos(\log x) + B\sin(\log x)$ b) $y = c_1e^{2x} + c_2e^{2x}$

3) Form the differential equation of family of lines parallel to the line $2x + 3y + 4 = 0$.

4) Verify that : $xy = \log y + c$ is a solution of the differential equation $\frac{dy}{dx} = \frac{y^2}{1 - xy}$.

5) Verify that : $y = a + \frac{b}{x}$ is a solution of the differential equation $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$.

6) For the following differential equation, find the particular solution satisfying the given condition. $(e^y + 1)\cos x + e^y \sin x \frac{dy}{dx} = 0$, when $x = \frac{\pi}{6}$, $y = 0$.

7) Solve the following differential equations.

i) $\frac{dy}{dx} = -k$, where k is constant.

ii) For the following differential equations find the particular solution satisfying the given condition. $(x - y^2x)\, dx - (y + x^2y)dy = 0$ when $x = 2, y = 0$.

iii) $\cos^2(x - 2y) = 1 - 2\frac{dy}{dx}$

iv) $xy \frac{dy}{dx} = x^2 + 2y^2, y(1) = 0$

v) $\left(1 + 2e^{\frac{x}{y}}\right) + 2e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) \frac{dy}{dx} = 0$

vi) $\cos^2 x \,.\, \frac{dy}{dx} + y = \tan x$

vii) $(x + y) \frac{dy}{dx} = 1$

viii)Find the equation of the curve which passes through the point $\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$ having slope of the tangent to the curve at any point (x, y) on it is $-\frac{4x}{9y}$

8) The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number double in 1 hour, find the number of bacteria after $2\frac{1}{2}$ hour. [Take $\sqrt{2} = 1.414$]

9) A body cools according to Newton's law from 100°c to 60°c in 20 minutes. The temperature of the surrounding being 20°c. How long will it take to cool down to 30°c?

10) The rate of decay of certain substance is directly proportional to the amount present at that instant. Initially, there are 25 gm of certain substance and two hours later it is found that 9 gm are left. Find the amount left after one more hour.

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7. Probability Distributions

- **Discrete random variables :**

  Definition : A random variable is said to be discrete random variable if the number of its possible values is finite or countably infinite.

  Note : The values of a discrete random variable are obtained by counting.

- **Continuous random variable**

  Definition : Continuous random variable has uncountably infinite possible values and these values from an interval of real numbers.

  Note : The value of a continuous random variable is obtained by measurement.

- **Probability Mass Function (p.m.f.) :**

  The function P(X) is said to be probability mass function if it satisfies,

  i) $pi \geq 0$, for all i and

  ii) $\Sigma pi = 1$.

- **Cumulative Distribution Function (c.d.f.) :**

  Definition : The cumulative distribution function (c.d.f.) of the discrete random variable X is denoted by F and is defined as follows. $F(x) = \mathrm{P}[X \leq x]$

- **Expected value :** $E(X) = \mu = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3 + \ldots x_n p_n$
- **Variance :** $\sigma_x^2 = Var(x) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \sum_{i=1}^{n} (x_i)^2 pi - (\sum_{i=1}^{n} x_i p_i)^2$
- **Standard Deviation :** $S.D(x) = x = \sigma n = \sqrt{Var(x)}$

  Probability Distribution of a Continuous Random Variable :

  Probability Density Function (p.d.f.) :

  A non-negative integrable function $f(x)$ is called the probability density function (p.d.f.) of X if it satisfies the following conditions.

  i) $f(x) \geq 0$, for all $x \in S$. ii) $\int_S f(x)\, dx = 1$.

  Also : $\mathrm{P}[a < X < b] = P[a < X \leq b] = P[a \leq X < b] = P[a \leq X < b] = \int_a^b f(x)\, dx$

- **Definition :** The Cumulative Distribution Function (c.d.f.) of a continuous random variable X is defined as $F(x) = \int_a^x f(t)\, dt$ for $a < x < b$.

Ex.1) Find the probability distribution of the number of sixes in two tosses of fair die.

Solution :

The sample space of the experiment is

S = {(1,1),(1,2),(1,3), ..., (1,6)(2,1),(2,2),(2,3), ..., (2,6) ... ... (6,1),(6,2),(6,3),...,(6,6)}

$\therefore\ n(s) = 36$

Let X denote the number of sixes.

$\therefore\ X = 0,1,2$

The probability distribution of X is as follows :

X	0	1	2
P $(X=x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

Ex.2) The probability distribution of X is as follows.

X	0	1	2	3	4
P $(X=x)$	0.1	k	2k	2k	k

Find i) k ii) P [X<2] iii) P[X≥3] iv) P [1≤X<4]

Solution :

The table gives a probability distribution and therefore

$P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3] + P[X = 4] = 1.$

$\therefore\ 0.1+k+2k+2k+k=1$

$6k = 0.9$

$\therefore\ k = 0.15$

i) $k = 0.15$

ii) $P[X<2] = P[X = 0] + P[X = 1] = 0.1 + k = 0.1 + 0.15 = 0.25$

iii) $P[X \geq 3] = P[X = 3] + P[X = 4] = 2k + k = 3(0.15) = 0.45$

iv) $P[1 \leq X<4]=P[X=1]+P[X = 2]+P[X=3]$

$= k + 2k + 2k = 5k = 5(0.15) = 0.75$

Ex.3) The probability distribution of X is as follows.

X	0	1	2	3	4
P ($X=x$)	0.2	0.3	0.25	0.15	0.1

Find the c.d.f. of X, F(2) and F(3).

Solution :

$F(2) = P[X \leq 2] = P[X=0] + P[X=1] + P[X=2]$

$= 0.2 + 0.3 + 0.25 = 0.75$

$F(3) = P[X \leq 3] = P[X=0] + P[X=1] + P[X=2] + P[X=3]$

$= 0.2 + 0.3 + 0.25 + 0.15 = 0.90$

Ex.4) Three coins are tossed simultaneously, X is the number of heads. Find expected value and variance of X.

Solution :

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT} and

X = {0, 1, 2, 3}

X = x_i	P = p_i	$x_i p_i$	$x_i^2 p_i$
0	$\frac{1}{8}$	0	0
1	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{12}{8}$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{9}{8}$
		$\sum_{i=0}^{3} x_i p_i = \frac{12}{8}$	$\sum_{i=1}^{3} x_i^2 p_i = \frac{24}{8}$

Then $E(X) = \sum_{i=0}^{3} x_i p_i = \frac{12}{8} = 1.5$

$Var(X) = \left(\sum_{i=1}^{n} x_i^2 p_i\right) - \left(\sum_{i=0}^{3} x_i^2 p_i^2\right)^2$

$= \frac{24}{8} - (1.5)^2 = 3 - 2.25 = 0.75$

Ex. 5) Find the mean and variance of the number randomly selected from 1 to 15.

Solution :

Let X denote the number selected.

Then X = 1,2,3.............15.

Each number selected is equi-probable therefore

$$P(1) = P(2) = P(3) = \ldots\ldots = P(15) = \frac{1}{15}$$

$$\mu = E(X) = \sum_{i=1}^{n} xi^2pi = 1\times\frac{1}{15} + 2\times\frac{1}{15} + 3\times\frac{1}{15} \ldots\ldots \ldots\ldots + 15)\times\frac{1}{15}$$

$$= (1+2+3+\ldots\ldots \ldots\ldots \ldots\ldots +15)\times\frac{1}{15} = \frac{15\times 16}{2}\times\frac{1}{15} = 8$$

$$var\ (X)=\left(\sum_{i=1}^{n} x^2ipi\right) - \left(\sum_{i=1}^{n} xipi\right)^2$$

$$= \left(1^2\times\frac{1}{15} + 2^2\times\frac{1}{15} + 3^2\times\frac{1}{15} \ldots\ldots \ldots\ldots + 15^2\times\frac{1}{15}\right) - (8)^2$$

$$= (1^2+2^2+3^2+ \ldots\ldots \ldots\ldots +15^2)\times\frac{1}{15} - (8)^2$$

$$= \frac{15\times 16\times 31}{6}\times\frac{1}{15} - (8)^2 = 82.67 - 64 = 18.67$$

Ex.6) *X* has the following probability density function.

$f(x)=\frac{x^3}{4}$ for $0 < x < 4$. What is the cumulative distribution function *X*?

Solution :

$$F(x) = \int_0^x f(x)\ dx$$

$$= \int_0^x \frac{x^3}{4}\ dx$$

$$= \frac{1}{4}\left[\frac{x^4}{4}\right]_0^x = \frac{1}{16}\ [x^4-0] = \frac{x^4}{16}$$

Ex.7) Find the c.d.f. *F*(*x*) associated with p.d.f. of r.v. X, where.

$$f(n)= \begin{cases} 3(1-2n^2)\ ; & 0<n<1 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Solution :

Since $f(x)$ is p.d.f. of r.v., c.d.f is

$$F(x) = \int_0^3 3(1-2x^2)\,dx=\left[3\left(x-\frac{2x^3}{3}\right)\right]_0^x = 3x-2x^3$$

Ex.8) **Find k if the following function is the p.d.f. of r.v. X.**

$f(x) = kx^2(1-x)$, for $0 < x < 1$

$= 0$, otherwise.

Solution :

Since $f(x)$ is the p.d.f. of r.v. X

$$\int_0^1 f(x)\, dx = 0$$

$$\therefore \int_0^1 kx^2(1-x)\, dx = 1$$

$$\therefore k \int_0^1 x^2 - x^3\, dx = 1$$

$$\therefore k \left[\frac{n^4}{3} - \frac{x^4}{4} \right]_0^1 = 1$$

$$\therefore k \left[\frac{1}{3} - \frac{1}{4} \right] = 1 \qquad \therefore k \left[\frac{1}{12} \right] = 1 \qquad \therefore k = 12$$

Ex.9) **For each of the following p.d.f. of r.v. X find (a) $P(X<1)$ (b) $P(|X|<1)$**

i) $f(n) = \begin{cases} \frac{n^2}{18}; & \text{for} -3 < n < 3 \\ 0; & \text{otherwise} \end{cases}$

ii) $f(n) = \begin{cases} \frac{x+2}{18}; & \text{for} -2 < x < 4 \\ 0; & \text{otherwise} \end{cases}$

Solution :

i) a) $P(X<1) = \int_{-3}^{1} \frac{x^2}{18} dx = \left[\frac{1}{18} \frac{x^3}{3} \right]_{-3}^{1} = \frac{1}{54}[1+27] = \frac{28}{54} = \frac{14}{27}$

b) $P(|X|<1) = P(-1<X<1) = \int_{-1}^{1} \frac{x^2}{18} dx = \left[\frac{1}{18} \frac{x^3}{3} \right]_{-1}^{1} = \frac{1}{54}[1+1] = \frac{2}{54} = \frac{1}{27}$

ii) a) $P(X<1) = \int_{-2}^{1} \frac{x+2}{18} dx = \frac{1}{18} \left[\frac{x^2}{2} + 2x \right]_{-2}^{1} = \frac{1}{18} \left[\frac{5}{2} + 2 \right] = \frac{1}{4}$

b) $P(|X|<1) = P(-1<X<1) = \int_{-1}^{1} \frac{x+2}{18}$

$dx = \frac{1}{18} \left[\frac{x^2}{2} + 2x \right]_{-1}^{1} = \frac{1}{18} = \left[\frac{5}{2} + \frac{3}{2} \right] = \frac{2}{9}$

Exercise 7.1

1) Find the probability distribution of number of heads in two tosses of coin.

2) Two dice are thrown simultaneously. If X denotes the number of sixes, find expectation of X

3) Find the expected value and variance of r.v. X whose p.m.f. are given below.

X	1	2	3
P (X = x)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

4) Find k if the following function represents the p.d.f. of ar.v.X.

$$f(n) = \begin{cases} kn(1-n) \ ; \text{ for } 0 < x < 1 \\ 0 \ ; \text{ otherwise} \end{cases} \quad \text{Also find, } P\left(X < \frac{1}{2}\right).$$

5) Find k if the following function represents the p.d.f. of ar.v. X.

$$f(n) = \begin{cases} kn \ ; \text{ for } 0 < n < 2 \\ 0 \ ; \text{ otherwise} \end{cases} \quad \text{Also find } P(1<X<2)$$

6) A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P (X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

Determine : i) k ii) $P(X>6)$ iii) $P(0<X<3)$

7) The probability distribution of discrete r.v. X is as follows.

X	1	2	3	4	5	6
P (X = x)	k	2k	2k	3k	k^2	$2k^2$

8) Find expected value and variance of X, where X is number obtained on uppermost face when a fair die is thrown.

9) In a meeting,70% of the members favor and 30% oppose a certain proposal. A member is selected at random and we take $X=0$ if he opposed, and $X=1$ if he is in favor. Find E(X) and Var (X)

10) The following is the p.d.f of a r.v. X : $f(x) = \begin{cases} \frac{x}{8} ; \text{ for } 0<x<4 \\ 0 \ ; \text{ otherwise} \end{cases}$

Find (i) $P(1<x<2)$ ii) $P(x>2)$.

11) Following is the p.d.f. of a continuous r.v. X.

$$f(x) = \begin{cases} \frac{x}{4} ; \text{ for } 0<x<4 \\ 0 \ ; \text{ otherwise} \end{cases}$$

i) Find the expression for the c.d.f of X. ii) Find $F(x)$ at $x = 0.5, 1.7$ and 5.

8. Binomial Distribution

Bernoulli Trial :

Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

i) Each trial has exactly two outcomes : success or failure.

ii) The probability of success remains the same in each trial.

Binomial distribution :

A random variable X is defined to have a Binomial distribution if p.m.f. is given by $P(X = x) = P(n) = n_{C_x}\, p^x q^{n-x}$ where $x = 0, 1, 2, 3, \ldots n$. Where n is number of trials, p is probability of success. Here $0 < p < 1$. q is probability of failure and $p + q = 1$. It is denoted by $X \sim B(n, p)$.

Also Mean $= \mu = E(X) = np$, Variance $= V(X) = npq$, S.D. $= \sigma = \sqrt{V(X)}$

Ex. 1) A die is thrown 6 times. If 'getting an odd number' is a success, find the probability of (i) 5 successes (ii) at least 5 successes (iii) at most 5 successes.

Solution : Here n = 6

Let X = number of successes = getting an odd numbers.

p = the probability of getting an odd number in a single throw of die

$= \frac{3}{6} = \frac{1}{2}$,

$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Here $X \sim B(6, \frac{1}{2})$. We know that $P(X = x) = p(x) = n_{C_x}\, p^x q^{n-x}$

i) $P(5) = 6_{C_5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^1 = 6\ \frac{1}{64} = \frac{3}{32}$

ii) $P(X \geq 5) = P(5) + P(6) = 6_{C_5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^1 + 6_{C_6}\left(\frac{1}{2}\right)^6 = \frac{3}{32} + \frac{1}{64}$

$= \frac{6+1}{64} = \frac{7}{64}$

iii) $P(X \leq 5) = 1 - P(X > 5) = 1 - 6_{C_6}\left(\frac{1}{2}\right)^6 = 1 - \frac{1}{64} = \frac{63}{64}$

Ex. 2) A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two successes.

Solution : Here $n = 4$.

Let X = number of doublets.

p = the probability of getting a doublet when a pair of dice is thrown

$= \frac{6}{36} = \frac{1}{6}$

$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

Here $X \sim B(4, \frac{1}{6})$. We know that $P(X = x) = p(x) = n_{C_x} p^x q^{n-x}$

$$P(X = 2) = 4_{C_2}\left(\frac{1}{2}\right)^2\left(\frac{5}{6}\right)^{(4-2)} = \frac{4!}{2!.2!} \cdot \frac{1}{36} \cdot \frac{25}{36} = \frac{25}{216}$$

Ex. 3) There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution : $n = 10$. Let X = number of defective items.

p = the probability of getting defective items $= \frac{5}{100} = \frac{1}{20}$,

$q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$,

Here $X \sim B(10, \frac{1}{20})$. We know that $P(X = x) = p(x) = n_{C_x} p^x q^{n-x}$

$$P(X \leq 1) = P(0) + P(1) = 10_{C_o}\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^{10-0} + 10_{C_1}\left(\frac{1}{20}\right)^1\left(\frac{19}{20}\right)^{10-1}$$

$$= 1.1.\left(\frac{19}{20}\right)^{10} + 10.\left(\frac{1}{20}\right)\left(\frac{19}{20}\right)^9 = \left(\frac{19}{20}\right)^9\left(\frac{19}{20} + \frac{10}{20}\right) = \frac{29}{20} \cdot \left(\frac{19}{20}\right)^9$$

Ex. 4) The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs (i) none (ii) not more than one (iii) more than one (iv) at least one will fuse after 150 days of use.

Solution : Here n = 5. Let X = number of fuse bulbs.

p = the probability of getting defective item = 0.05

$q = 1 - p = 1 - 0.05 = 0.95$.

Here $X \sim B(5, 0.05)$. We know that $P(X = x) = P(x) = n_{C_x} p^x q^{n-x}$

i) $P(X = 0) = P(u) = 5_{C_0} (0.05)^0 (0.95)^{5-0} = 1.1.(0.95)^5 = (0.95)^5$

ii) $P(X \leq 1) = P(0) + P(1) = 5_{C_0} (0.05)^0 (0.95)^{5-0} + 5_{C_1} (0.05)^1 (0.95)^{5-1}$

$= 1.1.(0.95)^5 + 5\ (0.05)^1 (0.95)^4 = (0.95)^4 (0.95 + 0.25) = 1.2\ (0.95)^4$

iii) $P(X > 1) = 1 - P(X \leq 1) = 1 - 1.2(0.95)^4$

iv) $P(X \geq 1) = 1 - P(0) = 1 - (0.95)^5$

Ex. 5) Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Solution : Here n = 6. Let X = number of sixes.

p = the probability that a die shows 6 on a single throw $= \frac{1}{6}$

$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

Here $X \sim B(6, \frac{1}{6})$. We know that $P(X = x) = p(x) = n_{C_x} p^x q^{n-x}$

$P(X \leq 2) = p(0) + p(1) + p(2)$

$= 6_{C_0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^6 + 6_{C_1} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^{6-1} + 6_{C_2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{6-2}$

$= 1.\ 1.\ \left(\frac{5}{6}\right)^6 + 6.\left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5 + \frac{6!}{2!.4!} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4 = \left(\frac{5}{6}\right)^6 + \left(\frac{5}{6}\right)^5 + \frac{6.5}{2.1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4$

$= \left(\frac{5}{6}\right)^5 \left[\left(\frac{1}{6}\right)^1 + 1 + \frac{3}{6}\right] = \left(\frac{5}{6}\right)^5 \left[\frac{5 + 6 + 3}{6}\right] = \left(\frac{5}{6}\right)^5 \frac{14}{6} = \frac{7}{3} \left(\frac{5}{6}\right)^5$

Ex. 6) Given that $X \sim B\ (n, p)$:

i) If $n = 10$ and $p = 0.4$, find $E\ (X)$ and Var (X)

ii) If $p = 0.6$ and $E(X) = 6$, find n and Var (X)

iii) If $n = 25$, $E(X) = 10$ find p and $SD(X)$.

Solution : $E(X) = np$, $V(X) = npq$, $SD(X) = \sqrt{V(X)}$

i) $q = 1 - p = 1 - 0.4 = 0.6$

$E(X) = np = 10 \times 0.4 = 4.$

$V(X) = npq = 10 \times 0.4 \times 0.6 = 2.4.$

ii) $E(X) = 6$

$\therefore\ np = 6$

$\therefore$ n $\times$ 0.6 = 6

$\therefore$ n = 10.q = 1 − p = 1-0.6 = 0.4

$V(X) = npq = 10 \times 0.6 \times 0.4 = 2.4.$

iii) $E(X) = 10$

$\therefore np = 10 = 25p$

$\therefore 25p = 10$

$\therefore p = \frac{10}{25} = \frac{2}{5}$ and $q = 1 - p = 1 - \frac{2}{5} = \frac{5-2}{5} = \frac{3}{5}$

$V(X) = npq = 10 \times \frac{3}{5} = 6$

$SD\,(X) = \sqrt{V(X)} = \sqrt{6}$

Exercise 8.1

1) If a fair coin is tossed 10 times, find the probability of getting
   i) exactly six heads ii) at least six heads iii) at most six heads

2) Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

3) Let the p.m.f. of r.v. X be $P(X = x) = {}^4C_x \left(\frac{5}{9}\right)^x \left(\frac{4}{9}\right)^{4-x}$, for $x = 0, 1, 2, 3, 4$ then find $E(X)$ and $V(X)$.

4) If $E(X) = 6$ and Var $(X) = 4.2$, find n and p.

5) Let $X \sim B\,(10, 0.2)$, Find (i) $P(X = 1)$ (ii) $P(X \geq 1)$ (iii) $P(X \leq 8)$

6) Let $X \sim B\,(n, p)$ (i) If $n = 10$, $E(X) = 5$, find p and Var (X).
   (ii) If $E\,(X) = 5$ and Var $(X) = 2.5$, find n and p.

7) If fair coin is tossed 10 times find the probability that it shows heads
   (i) 5 times (ii) in the first four tosses and tail in last six tosses.

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Answer Key (Part - II)

Exercise 1.1

1) $\dfrac{x}{\sqrt{x^2 + 5}}$ 2) $\dfrac{\cos (log\ x)}{-x}$ 3) $\dfrac{5x^4}{x^5 + 4}$

4) $-3x.5^{3\cos x}.log\ 5$ 5) $\dfrac{3x^2.\cos x^3}{\sqrt{2 \sin^3}}$ 6) $-5x^4 \tan (x)^5$

7) $3 + \dfrac{2}{3x - 4} - \dfrac{2}{3(2x + 5)}$ 8) $\dfrac{3}{2} \operatorname{cosec} \left(\dfrac{3x}{2}\right)$

Exercise 1.2

1) $\dfrac{3x^2}{\sqrt{1 - x^6}}$ 2) $\dfrac{2x}{2 + x^4}$ 3) $4x^3$ 4) 3

5) $\dfrac{1}{2}$ 6) $\dfrac{3}{2}$ 7) $\dfrac{1}{2}$ 8) 1

9) $\dfrac{2}{1 + x^2}$ 10) $\dfrac{3}{1 - x^2}$ 11) $\dfrac{2e^x}{1 + 2e^x}$ 12) $\dfrac{6}{1 + 9x^2}$

13) $\dfrac{7}{1 + (7x)^2} - \dfrac{3}{1 + (3x)^2}$ 14) $\dfrac{4}{1 + (4x)^2} - \dfrac{3}{1 + (3x)^2}$

Exercise 1.3

1) $\dfrac{e^{x2} (\tan x)^{\frac{x}{2}}}{(1 + x^2)^{\frac{x}{2}} \cos^3 x} \left[2x + x \operatorname{cosec}2x + \dfrac{1}{2} \log (\tan x) - \dfrac{3x}{1 + x^2} + 3 \tan x\right]$

2) $ax^{a-1} + a^x\log a + x^x [1 + \log x]$

3) $(\sin x)^{\tan x} [1 + \sec^2 x \log (\sin x)] - \dfrac{(2x^{\log x} \log x)}{x}$

Exercise 1.4

1) i) $\dfrac{5x^4 + 2xy + 3}{x^2 + 3xy^2 + 4y^2}$ ii) $\dfrac{2x + y \sin (xy) - \cos (x + y)}{3y^2 - x \sin (xy) + \cos (x + y)}$

iii) $\dfrac{2x (x + y) + ye^{xy} (x + y) - 1}{2y (x + y) - xe^{xy} (x + y) + 1}$

Exercise 1.5

1) i) $\frac{1}{t^2}$ ii) $\frac{2\sqrt{t}+1}{2\sqrt{t}-1}$ iii) $\frac{t \tan t}{\sin \log t}$ iv) $-\frac{1}{t}$

2) i) $\frac{3\sqrt{3}}{2}$ ii) $\frac{16}{3}$ iii) $\sqrt{3}$

Exercise 1.6

1) i) $6x + 14$ ii) $(x^2 + 4x + 2)e^x$ iii) $3 + 2 \log x$

2) i) $-\frac{2t(t^2-3)}{(1+t^2)^2}$ ii) $\frac{4\sqrt{2b}}{3a^2}$

Exercise 2.1

1) 0.8π cm^2/sec 2) i) valid ii) valid iii) not valid iv) valid

3) increasing 4) $y = 0$, $y = 4$ 5) $a = 2$, $b = -7$

6) 6 cm^3/sec 7) 3 km/hr 8) 0.9 m/s

9) i) 3.03704 ii) 0.8747 iii) 0.7859 iv) 9.098874 v) 4.6152

10) $c = \frac{5}{2} \in (1, 4)$ 11) $c = \frac{1}{2} \in (0, 1)$ 12) $x < -3$, $x > 8$

13) 15, 15 14) maximum height is 12.8 15) $x = 75$, $P = 4000$

Exercise 3.1

1) $ex - \frac{e^{-3x}}{3} + c$ 2) $\log x - x + c$ 3) $\tan x - \cot x - 4x + c$

4) $\frac{\left(\frac{3}{5}\right)^x}{\log\left(\frac{3}{5}\right)} - \frac{\left(\frac{4}{5}\right)^x}{\log\left(\frac{4}{5}\right)} + c$ 5) $\sec x - \tan x + x + c$

6) $\frac{3}{12} \sin 3x + \frac{3}{4} \sin x + c$ 7) $\sin x - \cos x + c$

8) $\frac{\pi}{4} x - \frac{x^2}{4} + c$

Exercise 3.2

1) $\log(\sec(\log x)) + c$ 2) $2\sqrt{\log x} + c$ 3) $\dfrac{\sec^7 x}{7} + c$

4) $\dfrac{1}{4}\log(x^4 + 1) + c$ 5) $\log(\sec(x^{ex}) + \tan(x^{ex})) + c$

6) $\cos(a - b)\log\sec(x + b) + x\sin(a - b) + c$

7) $\log(\log x) + c$ 8) $2\log(\sqrt{x} - 1) + c$

Exercise 3.3

1) $\dfrac{1}{5}\tan^{-1}\dfrac{x}{5} + c$ 2) $\dfrac{1}{4\sqrt{3}}\log\dfrac{2x - \sqrt{3}}{2x + \sqrt{3}} + c$

3) $\dfrac{1}{\sqrt{2}}\log\left(x + \sqrt{x^2 - \dfrac{5}{2}}\right) + c$ 4) $\dfrac{1}{4}\log\left(\dfrac{x - 6}{x - 2}\right) + c$

5) $\log\left(x - \dfrac{1}{2}\right) + \sqrt{x^2 - x - 6} + c$ 6) $9\sin^{-1}\dfrac{x}{9} + \sqrt{81 - x^2} + c$

7) $\dfrac{1}{2\sqrt{7}}\tan^{-1}\dfrac{2\tan x}{\sqrt{7}} + c$ 8) $\dfrac{1}{2\sqrt{3}}\log\left|\dfrac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right| + c$

9) $\dfrac{1}{2\sqrt{10}}\tan^{-1}\dfrac{\sqrt{5}\tan x}{2\sqrt{2}} + c$ 10) $\dfrac{1}{\sqrt{5}}\tan^{-1}\left(\dfrac{3\tan^{-1}\dfrac{x}{2} + 2}{\sqrt{5}}\right) + c$

11) $\dfrac{3}{4\sqrt{2}}\log\left(\dfrac{2\sqrt{2}\sin x + \sqrt{2} - 2}{2\sqrt{2}\sin x + \sqrt{2} + 2}\right) + c$

Exercise 3.4

1) $\dfrac{x^3}{9}(3\log x - 1) + c$ 2) $-\dfrac{x^2}{3}\cos(3x) + \dfrac{2}{9}x\sin(3x) + \dfrac{2}{27}\cos(3x) + c$

3) $\dfrac{x^2}{3}\tan^{-1}x - \dfrac{1}{2}(x - \tan^{-1}x)$ 4) $\dfrac{e^{2x}}{29}(2\sin(5x) - 5\cos(3x)) + c$

5) $\dfrac{1}{2}\log(\sec x + \tan x) + \dfrac{1}{2}\sec x\tan x + c$

6) $x^x + c$ 7) $e^x\tan x + c$ 8) $\dfrac{e^x}{x}$ 9) $x\sin(\log x) + c$

Exercise 3.5

1) $\frac{1}{4}\log(x-1)-2\log(x+2)+\frac{11}{4}\log(x+3)+c$

2) $\frac{1}{18}\log(3x-1)+\frac{1}{2}\log(x-1)-\frac{4}{9}\log(3x-2)+c$

3) $x-\log(x+3)+\log(x-2)+c$

Exercise 4.1

1) 1 2) $\frac{2}{3}$ 3) $\frac{4}{7\sqrt{2}}$ 4) 1

5) 2 6) $\frac{1}{2}$ 7) $\frac{1}{3}[\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{2}{3}$

8) $\frac{\pi}{6}$ 9) $\frac{\pi}{4}-\frac{1}{2}$ 10) $\sin(\log 3)$ 11) $e-\sqrt{e}$

12) $\frac{\pi}{4}-\frac{1}{2}\log 2$ 13) $\frac{1}{4}\log\left(\frac{2\sqrt{2}+1}{2\sqrt{2}-2}\right)$

14) $\frac{2}{3}\tan^{-1}\frac{1}{3}$ 15) $\frac{1}{2}\log 2$ 16) $\frac{\pi}{4}$

17) $\frac{1}{ab}\left[\tan^{-1}\left(\frac{ae}{b}\right)+\tan^{-1}\left(\frac{a}{be}\right)\right]$ 18) $\log\left(\frac{4}{3}\right)$ 19) $e-2\frac{2}{\log 2}$

20) 0 21) $2-\frac{\pi}{2}$ 22) 0 23) $\frac{\pi}{4}$

24) $\frac{\pi}{8}\log 2$ 25) $\frac{9}{2}$ 26) $\frac{5}{3}$

Exercise 5.1

1) i) 16 sq. unit ii) 20 sq. unit iii) 1 sq. unit

iv) $\frac{128}{3}$ sq. unit v) 21 sq. unit vi) 5 sq. unit

2) 20 sq. unit 3) $\frac{32}{3}$ sq. unit 4) 20π sq. unit

5) 25π sq. unit 6) $\frac{1}{12}$ sq. unit

Exercise 6.1

1) a) Order = 2, Degree = 2 b) Order = 2, Degree = Not defined

2) a) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$ b) $\frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 10y = 0$

3) $3 \frac{dy}{dx} + 2 = 0$ 6) $(e^y + 1) \sin x = 1$

7) i) $y = -kx + c$ ii) $(1 + x^2)(1 - y^2) = 5$ iii) $x = \tan(x - 2y) +$

iv) $x^2 + y^2 = x^4$ v) $x + 2ye^{\frac{x}{y}} = c$ vi) $y.e^{\tan x} = e^{\tan x}(\tan x - 1) + C$

vii) $x + y + 1 = ce^y$ viii) $4x^2 + 9y^2 = 36$

8) 5656 9) 1 hours 10) $\frac{27}{5}$ gm

Exercise 7.1

1)

X	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

2) $\frac{1}{3}$ 3) $E(x) = \frac{11}{5}$, $Var(x) = \frac{14}{25}$

4) $k = 6$, $P\left(X < \frac{1}{2}\right) = \frac{1}{2}$ 5) $k - \frac{1}{2}$, $P(1 < X < 2) = \frac{3}{4}$

6) i) $\frac{1}{10}$ ii) $\frac{17}{100}$ iii) $\frac{3}{10}$ 7) i) $\frac{1}{21}$ ii) $\frac{10}{21}$ iii) $\frac{1}{7}$

8) $E(x) = \frac{7}{2}$, $Var(x) = \frac{35}{12}$ 9) $E(x) = \frac{7}{10} = 0.7$, $Var(x) = \frac{21}{100} = 0.21$

10) i) $\frac{3}{16}$ ii) $\frac{3}{4}$ 11) i) $F(x) = \frac{x^2}{16}$ ii) $F(0.5) = \frac{1}{64}$, $F(1.7) = 0.18$, $F(5) = 1$

Exercise 8.1

1) (i) $\frac{105}{100}$ (ii) $\frac{193}{512}$ (iii) $\frac{53}{64}$ 2) $1 - \left(\frac{9}{10}\right)^{10}$

3) $\frac{20}{09}, \frac{80}{81}$ 4) 20 5) i) 2 (0.8) 9 ii) 1 − (0.8) 10 iii) $1 - (8.2)(0.2)^9$

6) i) $\frac{1}{2}$, 2.5 (ii) 10, $\frac{1}{2}$ 7) i) $\frac{53}{256}$ (ii) $\frac{105}{512}$

Practice Question Paper - 1

Class: XII (Arts and Science)

Subject : Mathematics and Statistics (40)

Marks: 80 **Time :** 3 Hours

General Instructions : The Question Paper is divided into four sections.

1) **Section A :**

   Q. No. 1 Contains **Eight** multiple choice questions carrying **Two** marks each.

   Q. No. 2 Contains **Four** very short answer type questions carrying **One** mark each.

2) **Section B :**

   Q. No. 3 to Q. No. 14 : contains **Twelve** short type questions carrying **Two** marks each. (Attempt any Eight)

3) **Section C :**

   Q. No. 15 to Q. No. 26: contains **Twelve** short type questions carrying **Three** marks each. (Attempt any Eight)

4) **Section D :**

   Q. No. 27 to Q. No. 34: contains **Eight** long type question carrying Four marks each. (Attempt any Five)

5) Use of log table is allowed. Use of calculator is not allowed.
6) Figures to the right indicate full marks.
7) In case of MCQ only first attempt will be considered for evaluation.
8) In LPP only rough sketch of graph is expected. Graph paper is not necessary.
9) Start answer to each section on a new page.

Section : A

Q. 1) Select and write the most appropriate answer from the given alternative for each sub-question. (2 marks each) (16 Marks)

i) $\sim p \wedge (p \vee q) \equiv$ .....

A) $p \wedge q$ B) $p \leftrightarrow q$ C) $\sim p \wedge q$ D) $\sim(p \rightarrow q)$

ii) Principal value of $\cos^{-1}\left(-\frac{1}{2}\right) =$

A) $\frac{\pi}{3}$ B) $\frac{2\pi}{3}$ C) $-\frac{\pi}{3}$ D) $-\frac{2\pi}{3}$

iii) If $\overline{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\overline{b} = 3\hat{i} + m\hat{j} - 3\hat{k}$, and $\overline{a}$ is perpendicular to $\overline{b}$ then $m = ...$

A) 2 B) −2 C) 3 D) −3

iv) The equation of a line passing through (2, 4, −1) and parallel to the line $\frac{x}{3} = \frac{y-1}{2} = \frac{z+1}{4}$ is ....

A) $\frac{x+2}{3} = \frac{y+4}{2} = \frac{z-1}{4}$ B) $\frac{x-2}{3} = \frac{y-4}{2} = \frac{z+1}{4}$

C) $\frac{x-2}{-3} = \frac{y-4}{-2} = \frac{z-1}{-4}$ D) $\frac{x+2}{-3} = \frac{y+4}{-2} = \frac{z-1}{-4}$

v) The slope of the tangent to the curve $y = \frac{x}{x^2+2}$ at origin is .....

A) 2 B) 0 C) 1 D) −1

vi) The area of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is ....

A) 25π B) 20π C) 16π D) 1π

vii) $y = (c_1 + c_2 x)e^x$ is the solution of ......

A) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ B) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0$

C) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ D) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$

viii) If p.m.f. of r.v.X is $P(X = x) = \frac{4}{x}\left(\frac{5}{9}\right)^x \left(\frac{4}{9}\right)^{4-x}$, $x = 0, 1, 2, 3, 4$ then $Var(X)$ = ...

A) 0.1876 B) 0.7876 C) 0.0876 D)0.9876

Q. 2) Answer the following questions: (1 mark each) (4 Marks)

i) Write the negations of the statement : 'Some triangles are equilateral triangle'.

ii) Find the unit vector in the direction of $2\hat{i} - 2\hat{j} + \hat{k}$

iii) Write the integral of $\frac{f'(x)'}{\sqrt{(f(x)}}$ w.r.t.x

iv) Find the degree of the differential equation $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$

Section : B

Attempt any EIGHT of the following questions: (2 marks each) (16 Marks)

Q. 3) Construct the circuit for the statement $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim q \wedge r)$

Q. 4) Find the inverse of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ by using elementary row transformation

Q. 5) In ΔABC if $a = 13$, $b = 14$, $c = 15$, then find $\sin\left(\frac{A}{2}\right)$

Q. 6) Find k, if the sum of slopes of the lines given by $x^2 + kxy - 3y^2 = 0$ is twice their product.

Q. 7) If the vectors $3\hat{i} + 5\hat{k}$, $4\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} + 4\hat{k}$ are three co–terminus edges of the parallelepiped, then find the volume of the parallelepiped.

Q. 8) Find the angle between lines $\overline{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\overline{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$.

Q. 9) If $\sqrt{x} + \sqrt{y} = \sqrt{a}$, then find $\frac{dy}{dx}$

Q. 10) Evaluate $\int \frac{1}{x^2 + 8x + 12}\, dx$

Q. 11) If $\int_0^k \frac{1}{2 + 8x^2}\, dx = \frac{\pi}{16}$, then find the value of k.

Q. 12) Find the area of the region bounded by the parabola $y^2 = 16x$ and its latus rectum

Q. 13) Form the differential equation by eliminating arbitrary constants of the equation $y = Ae^{3x} + Be^{-3x}$.

Q. 14) Find $E(X)$, where X is a discrete random variable with p. m. f. given by

$X = x$	0	1	2	3
$P(X = x)$	0.1	0.2	0.4	0.3

Section : C

Attempt any EIGHT of the following questions: (3 marks each) (24 Marks)

Q. 15) Construct the truth tables for the statement pattern $(\sim p \vee q) \rightarrow [p \wedge (q \vee \sim q)]$. Interpret your results.

Q. 16) Find principal solutions of $\cos 5\theta = \sin 3\theta$

Q. 17) If A, B, C are the measures of angles of a ΔABC and a, b, c are lengths of sides BC, AC, AB respectively then prove that $b^2 = a^2 + c^2 - 2ac \cos B$

Q. 18) Find the direction cosines of the lines which is perpendicular to the lines with direction ratios −1, 2, 2 and 0, 2, 1.

Q. 19) Show that lines $\bar{r} = (2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$ and $\bar{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$ are coplanar.

Q. 20) Find the vector equation of the plane passing through points $A(1, 1, 2)$, $B(0, 2, 3)$ and $C(4, 5, 6)$.

Q. 21) If $e^x + e^y = e^{(x + y)}$ then show that $\frac{dy}{dx} = -e^{(y - x)}$

Q. 22) Find the approximate value of $\sqrt{8.95}$

Q. 23) Evaluate $\int \frac{2x}{4 - 3x - x^2}\, dx$

Q. 24) Find the general solution of the D.E. $3e^x \tan y\, dx + (1 + e^x) \sec^2 y\, dy = 0$

Q. 25) The p.m.f. of a r.v. X is given below.

X	0	1	2	3	4
$P(X = x)$	0.1	k	$2k$	$2k$	k

Find i) k, ii) $P(X < 2)$, iii) $P(X \geq 3)$

Q. 26) Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

Section – D

Attempt any FIVE of the following questions: (4 marks each) (20 Marks)

Q. 27) The sum of three numbers is 6. If we multiply third number by 3 and add it to the second number we get 11. By adding first and the third numbers we get a number which is double the second number. Use this information and find a system of linear equations. Find the three numbers using matrices.

Q. 28) Prove that homogeneous equation of degree two in x and y, $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin if $h^2 - ab \geq 0$.

Q. 29) If the points $A(3, 0, p)$, $B(-1, q, 3)$, $C(-3, 3, 0)$ are collinear , then find the ratio in which the point C divides the line segment AB. Also find the values of p and q.

Q. 30) Solve the following LPP by using graphical method,

Maximize $z = 11x + 8y$

Subject to, $x \leq 4,\ y \leq 6,\ x + y \leq 6,\ x \geq 0,\ y \geq 0$

Q. 31) If y = $f(x)$ is a differentiable function of x such that the inverse function $x = f^{-1}(y)$ is defined, then show that $\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}}$ where $\frac{dy}{dx} \neq 0$

Hence find the derivative of the inverse of the function $y = 2x^3 - 6x$.

Q. 32) If u and v are functions of x, then $\int uv\, dx \;= u\int v\, dx - \int \left(\frac{dy}{dx} \int v\, dx\right)\; dx$

Also, evaluate $\int x \sin x\, dx$

Q. 33) Verify Rolle 's theorem for the function $f(x) = x^2 - 4x + 10$ on $[0, 4]$

Q. 34) Evaluate $\int_0^1 \frac{x \tan^{-1} x}{(1 + x^2)^{\frac{3}{2}}}\, dx$

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Practice Question Paper - 2

Class: XII (Arts and Science)

Subject : Mathematics and Statistics (40)

Marks: 80 **Time :** 3 Hours

General Instructions : The Question Paper is divided into four sections.

1) **Section A :**

   Q. No. 1 Contains **Eight** multiple choice questions carrying **Two** marks each.

   Q. No. 2 Contains **Four** very short answer type questions carrying **One** mark each.

2) **Section B :**

   Q. No. 3 to Q. No. 14 : contains **Twelve** short type questions carrying **Two** marks each. (Attempt any Eight)

3) **Section C :**

   Q. No. 15 to Q. No. 26 : contains **Twelve** short type questions carrying **Three** marks each. (Attempt any Eight)

4) **Section D :**

   Q. No. 27 to Q. No. 34: contains **Eight** long type question carrying Four marks each. (Attempt any Five)

5) Use of log table is allowed. Use of calculator is not allowed.

6) Figures to the right indicate full marks.

7) In case of MCQ only first attempt will be considered for evaluation.

8) In LPP only rough sketch of graph is expected. Graph paper is not necessary.

9) Start answer to each section on a new page.

Section : A

Q. 1) Select and write the most appropriate answer from the given alternatives for each sub-question : (2 marks each) (16 Marks)

i) The negation of : 'For every natural number x, $x + 5 > 4$' is ….

A) $\forall\ x \in N, x + 5 < 4$

B) $\forall\ x \in N, x - 5 < 4$

C) For every integer x, $x + 5 < 4$

D) there exist a natural x, for which $x + 5 \leq 4$

ii) The principal solutions of $\sec x = \frac{2}{\sqrt{3}}$ are ....

A) $\frac{\pi}{3}, \frac{11\pi}{6}$ B) $\frac{\pi}{6}, \frac{11\pi}{6}$

C) $\frac{\pi}{4}, \frac{11\pi}{4}$ (ϖ)/(4),(11ϖ)/4 D) $\frac{\pi}{6}, \frac{11\pi}{4}$

iii) Which of the following represents direction cosines of a line ?

A) $0, \frac{1}{\sqrt{2}}, \frac{1}{2}$ B) $0, -\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}$

C) $0, \frac{\sqrt{3}}{2}, \frac{1}{2}$ D) $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

iv) If the point A(λ, 5, –2) lies on the line $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$, then the value of λ is ....

A) –1 B) 1 C) 8 D) –8

v) The function $f(x) = x^3 - 3x^2 + 3x - 100$ is ....

A) increasing B) decreasing

C) strictly decreasing D) neither increasing nor decreasing

vi) The area of the region bounded by $y = x$, $x = 1$, $x = 2$ and the X-axis is .....

A) $\frac{3}{2}$ B) 2 C) $\frac{1}{2}$ D) 4

vii) The order and degree of the differential equation $\left(1 + \left(\frac{dy}{dx}\right)^3\right)^{\frac{2}{3}}$ respectively are ....

A) 3 , 1 B) 1 , 3 C) 3 , 3 D) 1 , 1

viii) For X ~ B (n, p), if V(x) = 2.4 and p = 0.4 then n = ...

A) 10 B) 20 C) 30 D) 40

Q. 2) Answer the following questions: (1 mark each) (4 Marks)

i) Write the dual of the statement: $(p \vee q) \wedge t$.

ii) Find k if $C(k, 4, -2)$ is the mid point of the segment joining $A(-2, 1, 0)$ and $B(2, 7, -4)$.

iii) If $f'(x) = x^{-1}$, then find $f(x)$.

iv) Find the integrating factor of linear differential equation $\frac{dy}{dx} + y \sec x = \tan x$.

Section : B

Attempt any EIGHT of the following questions: (2 marks each) (16 Marks)

Q. 3) Write the converse and contrapositive of the statement : "If a function is differentiable then it is continuous."

Q. 4) Find the cofactors of the elements of the matrix $\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$

Q. 5) Find the general solution of $\cos x = \sin x$

Q. 6) Find the value of k, if the $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$

Q. 7) Find the volume of the parallelepiped whose coterminous edges are :

$2\hat{i} + 3\hat{j} - 4\hat{k}, \quad 5\hat{i} + 7\hat{j} + 5\hat{k}, 4\hat{i} + 5\hat{j} - 2\hat{k}$

Q. 8) Find the vector equation of the line passing through A(3, 4, −7) and B (6, −1, 1)

Q. 9) If $y = log\,(\sec x + \tan x)$ then find $\frac{dy}{dx}$

Q. 10) Evaluate: $\int \frac{1}{x} \log x \, dx$

Q. 11) If $\int_0^k \frac{1}{2 + 8x^2} \, dx = \frac{\pi}{6}$, then find the value of k.

Q. 12) Find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 3$

Q. 13) Form the differential equation by eliminating arbitrary constants of the equation $y = Ae^{5x} + Be^{-5x}$.

Q. 14) Find the variance of r.v. X for the following probability distribution:

$X = x$	0	1	2	3	4
P $(X = x)$	0.2	0.4	0.2	0.1	0.1

Section : C

Attempt any EIGHT of the following questions: (3 marks each) (24 Marks)

Q. 15) Using truth table prove that $p \vee (q \wedge r) \equiv (p \vee q) \vee (p \vee r)$

Q. 16) In ΔABC, with usual notations prove that : $c = a \cos B + b \cos A$

Q. 17) Prove that $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

Q. 18) Using vectors prove that the line segments joining mid–points of adjacent sides of a quadrilateral form a parallelogram.

Q. 19) Find the angle between the planes $\bar{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 13$ and $\bar{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 31$

Q. 20) Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$

Q. 21) Solve the differential equation: $(x^2 + y^2)\ dx - 2xy\ dy = 0$

Q. 22) Prove that $\int \sqrt{a^2 - x^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right) + c$

Q. 23) Verify Rolle's Theorem for the function $f(x) = x^2 - 5x + 9,\quad x \in [1, 4]$

Q. 24) If $log\ (x + y) = log\ (xy) + p$, where p is constant then prove that $\frac{dy}{dx} = -\frac{y^2}{x^2}$

Q. 25) Given the p.d.f. of continuous random variable X as: $f(x) = \begin{cases} \frac{x^3}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

Determine the c.d.f. of X and hence find (i) $P(X < 1)$ (ii) $P(X \le -2)$

Q. 26) A fair coin is tossed 10 times. Find the probability of getting:

(i) exactly six heads (ii) at least six heads.

Section – D

Attempt any FIVE of the following questions: (4 marks each) (20 Marks)

Q. 27) Solve the following system of equations by using reduction method

$2x - y + z = 1,\quad x + 2y + 3z = 8,\quad 3x + y - 4z = 1$

Q. 28) Prove that the homogeneous equation of degree two in x and y i.e. $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin, if $h^2 - ab \ge 0$.

Q. 29) Using vectors prove that the altitudes of a triangle are concurrent.

Q. 30) Maximize $z = 6x + 4y$ subject to $x \le 2,\ x + y \le 3,\ -2x + y \le 1,\ x \ge 0,\ y \ge 0$

Q. 31) If y is a differentiable function of u and u is a differentiable function of x, then prove that the composite function y is a differentiable function of x and $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Hence find $\frac{dy}{dx}$ if $y = \sqrt{(x^3+7)}$

Q. 32) The perimeter of a triangle is 10 cm. If one of the sides is 4 cm. then what are the other sides of the triangle for its maximum area?

Q. 33) $\int \frac{3x^2-2}{(x^2-1)\ (x^2-2)}\, dx$

Q. 34) Prove that, $\int_{-a}^{a} f(x)\ dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(x) \text{ is an even function.} \\ 0 & \text{if } f(x) \text{ is an odd function.} \end{cases}$

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Std. XII - Subject : Mathematics and Statistics

Part - II

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