"Comprehensive Support for Students in Mathematics subject seeking to Overcome Past Setbacks."

MATHEMATICS AND STATISTICS

Std. - XII

(Commerce)

Part - I

State Council of Educational Research and Training, Maharashtra, Pune

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Specialized Mathematics Study Materials for HSC Students

Subject : Mathematics and Statistics (Commerce) Code : 88

OBJECTIVES OF THE BOOKLET

This booklet is prepared for the help of the students who will be appearing for the Supplementary Examination to be held in July 2024 and thereafter too. It is prepared as such students could not score the minimum score to pass in the written examination held in February 2024.

This booklet is designed to boost the confidence of the students. It will definitely help them to score good marks in the forthcoming examination. It will be a great support for the students who lack behind others.

It is prepared in a systematic and easiest way by the expert teachers. The students are aware of the textbook as well as the examination pattern (MCQ's, Fill in the Blanks, True/False, 3 Marks and 4 Marks, Activities questions). Still, this booklet elaborates every segment in detail. It considers the level of the students.

By studying as suggested in the booklet, we are quite sure that the students will be able to practice a lot with given guidelines. They will score and step into the world of success.

The main objectives can be summarized as under :

1) To facilitate the essential study material to the students to confidently face the HSC Board Examination.
2) To help every average and the below average student to achieve 100% success at the HSC Board Examination.
3) To motivate the below average students to score more than their expectation in the Mathematics Subject which they find as most difficult.
4) To include tools and exercises that allow students to evaluate their own progress and understand their improvement areas.
5) To help the teachers to reach out to students who struggle to pass in the Mathematics subject at the HSC Board Exam with the help of this material.
6) Each chapter in the booklet contains important concepts in short.
7) Based on these concepts simple solved examples are given.
8) Practice questions with hints and answers are given.
9) Two practice question papers will definitely help students.

INTRODUCTION

Dear Students,

It does not matter if you did not score well in the regular examination held in February 2024. Remember,"every setback is a setup for a comeback." Your previous attempt must have taught you something valuable. We believe in your potential to overcome this hurdle and excel in your upcoming exams.

After a comprehensive analysis of the results, SCERT, Pune has taken an initiative for the upliftment of students who could not achieve the minimum passing score. It was found that some fundamental concepts were not clear to the students. Hence, a significant effort was made to prepare this booklet.

This booklet is designed specifically for those who did not achieve the desired results in their previous Mathematics exam. We understand that facing a setback can be challenging, but it also presents an invaluable opportunity for growth and learning. Our goal with this booklet is to provide you with comprehensive resources and targeted exercises to help you strengthen your understanding of key mathematical concepts. We have carefully curated the content to address common areas of difficulty and to reinforce fundamental principles essential for success in Mathematics.

This booklet will help you to prepare for the supplementary examination to be held in July 2024. Through a combination of clear explanations, step-by-step problem-solving strategies, and ample practice questions, we aim to build your confidence and competence in the subject. Remember, perseverance and a positive mindset are crucial as you work through this material.

Use this booklet diligently, seek help when needed, and stay committed to your studies. With dedication and effort, you can turn this experience into a stepping stone toward academic success. This resource will also prove to be extremely useful for teachers as they assist students in preparing for the supplementary examination. It will boost your confidence to appear for the exam once again. New students in the coming years can also benefit from this booklet.

Best wishes for your journey ahead.

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Part - I

Part - I

1. Mathematical Logic

STATEMENT OR PROPOSITION :

A) A simple sentence which is assertive or declarative and which is true or false (but not both) is a statement or a proposition in logic.

B) Mathematical identities are considered to be statement.

C) Sentences which are interrogative, exclamatory, imperative or open are not statements.

Truth value of a statement : The truth or falsity of a statement is called its truth value. When a statement is **true,** its truth value is 'true' and is denoted by **'T'** when a statement is false, its truth value is 'false' and it is denoted by **'F'**.

Logical connectives : the words or group of words like 'and', 'or', 'not', 'if ..... then', 'if and only if' can be used to join two simple sentences. Such words or groups are 'logical connectives' or simply 'connectives'.

Compound statements : the statements which are formed by combining two or more simple statements using 'logical connectives' are called 'compound statements'.

Component statement : simple statements which are used in compound statements are called 'component statements'.

The truth value of a compound statement depends upon the truth value of its component statements and the connectives used.

Sr. No.	Connective	Symbol	Name of compound statement
1)	Not	$\sim$	Negation
2)	Or	$\wedge$	Disjunction
3)	And	$\vee$	Conjunction
4)	If .... then	$\rightarrow$	Implication
5)	If and only if (iff)	$\leftrightarrow$	Double Implication

- **NOTE :** 1) In negation true statement becomes false and false statements becomes true. In conjunction (and) if both the statements are true then final statement is true otherwise final statement is false. In disjunction (or) if both the statements are false then final statement is false otherwise final statement is true. In implication true statement implies false is false otherwise it is true. In double implication if both the statements are having same truth values then final statement is true otherwise it is false. Refer following truth tables. Here p and q are two statements that may be true or false and when we join them by connectives then we get 4 options as shown below.

p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	T	F
F	F	F	F	T	T

p	$\sim p$
T	F
F	T

2) If a statement pattern involve n component statements p,q,r,... and each of them has two possible truth values namely T and F, then the truth table of the statement pattern consists of 2^n rows.

3) If a statement pattern contains 'm' connectives and 'n' component statements then the truth table of the statement pattern consists of (m+n) columns.

Quantifier : The symbol '$\forall$' (for all) is known as universal quantifier.

The symbol '$\exists$' (there exist) is called existential quantifier.

An open sentence with a Quantifier becomes a statement, it is called a quantified statement.

Converse, Inverse, Contrapositive of an implication $p \rightarrow q$

Converse : $q \rightarrow p$ Inverse : $\sim p \rightarrow \sim q$ Contrapositive : $\sim q \rightarrow \sim p$

TAUTOLOGY, CONTRADICTION AND CONTINGENCY : A statement pattern which is always true irrespective of truth values of its component statement letters is called a tautology. In general, a tautology is denoted by 'T' or 't'.

A statement pattern which is always false irrespective of truth values of its component statement is called a contradiction. In general, a contradiction is denoted by 'F' or 'c'.

A statement pattern which is neither Tautology nor Contradiction is called Contingency.

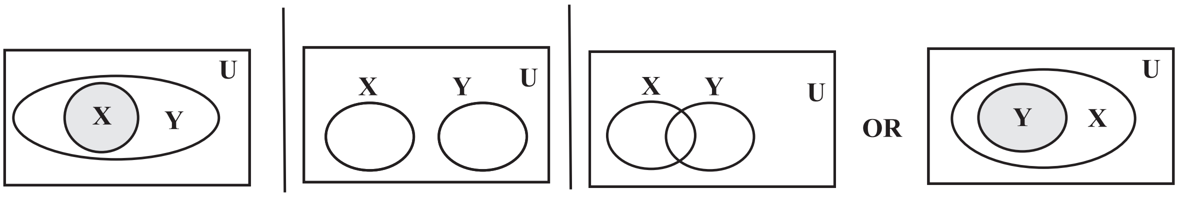
Duality : Let A be a statement pattern involving one or more connectives $\wedge, \vee, \sim$. Then, the dual of A is the statement pattern obtain from A by replacing '$\wedge$' by '$\vee$', '$\vee$' by '$\wedge$', 't' by 'c' and 'c' by 't'. We denote the dual of A by A'

$p \rightarrow q \equiv \sim p \vee q$ therefore dual of $p \rightarrow q$ is $\sim p \wedge q$

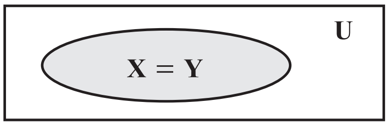
$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$ therefore dual of $p \leftrightarrow q$ is $(\sim p \wedge q) \vee (p \wedge \sim q)$

Venn Diagram : Representation of True statements

1) All x's are y's 2) No x's are y's 3) Some x's are y's



4. All X's are Y's and all Y's are X's



Algebra of statements : (Some standard equivalent statements using laws)

1)	Idempotent Laws	$p \vee q \equiv p$	$p \wedge p \equiv p$
2)	Commutative Laws	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
3)	Associative Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r) \equiv p \vee q \vee r$	$(p \vee q) \vee r \equiv p \vee (q \vee r) \equiv p \vee q \vee r$
4)	Distributive Laws	$(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \vee r)$	$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$
5)	De Morgan's Laws	$\sim (p \vee q) \equiv \sim q \wedge \sim p$	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
6)	Conditional Laws	$p \rightarrow q \equiv \sim q \rightarrow \sim p \equiv \sim p \vee q$	$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$
7)	Involution Laws	$\sim (\sim p) \equiv p$	$\sim T \equiv F, \ \sim F \equiv T$
8)	Complement Laws	$p \vee \sim p \equiv T$ (or t), $\sim t \equiv c$	$p \wedge \sim p \equiv F$ (or c), $\sim c \equiv t$
9)	Identity Laws	$p \vee F \equiv p$ or $p \vee c \equiv p$ $p \vee T \equiv T$ or $p \vee t \equiv t$	$p \wedge F \equiv F$ or $p \wedge c \equiv c$ $p \wedge T \equiv p$ or $p \wedge t \equiv p$

Solved Examples :

Q. 1) State which of the following sentences are statements. Justify.

1) March is a month in the English calendar.

Sol. : Statement (Simple sentence)

2) May god bless you!

Sol. : Not a statement (Imperative sentence)

3) The sum of interior angles of a quadrilateral is 360°.

Sol. : Statement (maths law and simple sentence)

4) Every natural number is a real number.

Sol. : Statement (Simple sentence)

5) Is the car yellow in colour?

Sol. : Not a statement (Introgative sentence)

6) Every quadratic equation has only one real root.

Sol. : Statement (Simple sentence)

7) -14 is a rational number.

Sol. : Statement (Simple sentence)

8) Please, come here.

Sol. : Not a statement (Request)

9) The sum of cube roots of unity is one.

Sol. : Statement (Simple sentence)

10) He is a bad person.

Sol. : Not a statement (Decision/opinion)

Q. 2) Express the following in symbolic form.

1) Banana is a fruit but potato is a vegetable.

Sol. : Let p : Banana is a fruit. and q : Potato is a vegetable.

$\therefore$ symbolic form of the statement is $p \wedge q$

2) We play kabaddi or go for cycling.

Sol. : Let p : We play kabaddi. and q : we go for cycling.

symbolic form of the statement is $p \vee q$

3) Seema stays at home while Reeta and Geeta go for a movie.

Sol. : Let p : Seema stays at home. and q : Reeta and Geeta go for a movie.

$\therefore$ symbolic form of the statement is $p \wedge q$

4) The drug is effective though it has side effects.

Sol. : Let p : The drug is effective. and q : The drug has side effects.

symbolic form of the statement is $p \wedge q$.

5) In spite of bad weather , India won the cricket match.

Sol. : Let p : The weather is bad . and q : India won the cricket match.

$\therefore$ symbolic form of the statement is $p \wedge q$.

6) If $68 = 2 \times 29$, then lions can drive the car.

Sol. : Let p : $68 = 2 \times 29$ and q : Lions can drive the car.

$\therefore$ symbolic form of the statement is $p \to q$.

7) $a^2+b^2=(a+b)^2$, iff $ab = 0$.

Sol : Let p : $a^2+b^2=(a+b)^2$ and q : $ab = 0$.

$\therefore$ symbolic form of the statement is $p \leftrightarrow q$

8) To be brave, is necessary and sufficient condition to climb the mount Everest.

Sol : Let p : To be brave and q : To climb the mount Everest.

$\therefore$ symbolic form of the statement is $p \leftrightarrow q$.

9) The necessary condition for the existence of a tangent to the curve so that the function is continuous.

Sol : Let p : The function is continuous. and q : A tangent to a curve exist.

$\therefore$ symbolic form of the statement is $p \to q$.

10) Neither Aman nor Raman are going for the picnic.

Sol : Let p : Aman is going for picnic and q : Raman is going for picnic.

$\therefore$ symbolic form of the statement is $\sim p \wedge \sim q$

Q.3) Write the truth values of the statements.

1) 2 is rational number or $4-3i$ is a complex number.

Sol : $T \vee T = T$

2) 16 is a perfect square but 17 is a prime number.

Sol : $T \wedge T = T$

3) Prepaid expense is an asset or accountancy is a part of book keeping.

Sol : $T \vee F = T$

4) Neither 21 is a prime number nor it is a divisible by 3.

Sol : $\sim F \wedge \sim T = T \wedge F = F$

5) It is not true that $3-7i$ is a real number.

Sol : $\sim F = T$

6) Jupiter is a planet and Mars is a star.

Sol : $T \wedge F = F$

7) 6 is an even number or Pune is a harbour.

Sol : $T \vee F = T$

8) If London is in England then $7 + 3 = 9$.

Sol : $T \rightarrow F = F$

9) SEBI office is in Mumbai or Dalal street is in London.

Sol : $T \vee F = T$

10) Every accountant is free to apply his own accounting rule iff (if and only if) machinery is an asset.

Sol : $F \leftrightarrow T = F$

Q. 4) If p, q are true statements and r , s are false statements then find the truth values of the following statements.

1) $(\sim p \vee q) \rightarrow (r \wedge \sim s)$

Sol. $(\sim p \vee q) \rightarrow (r \wedge \sim s)$

$\equiv (\sim T \vee T) \rightarrow (F \wedge \sim F)$

$= (F \vee T) \rightarrow (F \wedge T)$

$= T \rightarrow F$

$= F$

2) $(p \leftrightarrow \sim q) \wedge (\sim r \vee s)$

Sol. $(p \leftrightarrow \sim q) \wedge (\sim r \vee s)$

$\equiv (T \leftrightarrow \sim T) \wedge (\sim F \vee F)$

$= (T \leftrightarrow F) \wedge (T \vee F)$

$= F \wedge T$

3) $[(p \rightarrow \sim q) \leftrightarrow \sim s] \vee r$

Sol. $[(p \rightarrow \sim q) \leftrightarrow \sim s] \vee r$

$\equiv [(T \rightarrow \sim T) \leftrightarrow \sim F] \vee F$

$= [(T \rightarrow F) \leftrightarrow T] \vee F$

$= [F \leftrightarrow T] \vee F$

$= F \vee F = F$

4) $[(r \wedge \sim s) \vee q] \rightarrow (p \leftrightarrow r)$

Sol. : $[(r \wedge \sim s) \vee q] \rightarrow (p \leftrightarrow r)$

$\equiv [(F \wedge \sim F) \vee T] \rightarrow (T \leftrightarrow F)$

$= [(F \wedge T) \vee T] \rightarrow (T \leftrightarrow F)$

$= [F \vee T] \rightarrow (T \leftrightarrow F)$

$= T \rightarrow F = F$

Q. 5) If B = { 5 , 6 , 8 , 10 } , determine the truth value of the following :

1) $\exists\, x \in B$, such that $3x+4=28$

Sol : $3x = 24 \rightarrow x=8 \in B$ $\therefore$ True

2) $\forall\, x \in B$, $x+7 < 14$

Sol : $x < 7$ but all x in B are not less than 7 $\therefore$ False

3) $\forall\, x \in B$, $4x - 3 \geq 17$

Sol : $4x \geq 20$, $x \geq 5$ for all $x \in B$ $\therefore$ True

4) $\exists\, x \in B$, such that x is even.

Sol : True

5) $\exists\, y \in B$, such that $(y - 10) \in N$

Sol : $\therefore$ False

Q. 6) Use quantifiers to convert each of the following open sentences defined on N, into a true statements.

1) $x^2 = 36$

Sol. : $\exists\, x \in N$, such that $x^2 = 36$

2) $5x - 3 < 10$

Sol. : $\exists\, x \in N$, such that $5x - 3 < 10$

3) $x - 7 = 9$

Sol. : $\exists\, x \in N$, such that $x - 7 = 9$

4) $y^2 + 3 \leq 7$

Sol. : $x\ N$, such that $y^2 + 3 \leq 7$

5) $x^2 \geq 1$

Sol. : $\forall\, x \in N$, $x^2 \geq 1$

6) $x > 0$

Sol. : $\forall\, x \in N$, $x > 0$

7) $y^2 - 11y + 30 = 0$

Sol. : $\exists\, x \in N$, such that $y^2 - 11y + 30 = 0$

8) $x = 0$

Sol. : $\nexists\, x \in N$ such that $x = 0$

Q. 7) State the Converse, Inverse and Contra positive of the following statements.

i) If a quadrilateral is a square then it is not a rhombus.

Solution : Let p : A quadrilateral is a square and q : A quadrilateral is not a rhombus.

Converse : $q \rightarrow p$: If a quadrilateral is not a rhombus then it is a square.

Inverse : $\sim p \rightarrow \sim q$: If a quadrilateral is not a square then it is a rhombus.

Contra positive : $\sim q \rightarrow \sim p$: A quadrilateral is a rhombus then it is not a square.

ii) If an integer is not divisible by 2 then it is odd.

Solution : Converse : If an integer is odd then it is not divisible by 2.

Inverse : If an integer is divisible by 2 then it is not odd.

Contra positive : If an integer is not odd then it is divisible by 2.

Q. 8) Prepare the truth table of the following statement pattern. $(p \wedge q) \rightarrow \sim P$

Solution :

p	q	$\sim P$	$p \wedge q$	$(p \wedge q) \rightarrow \sim P$
T	T	F	T	F
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Q. 9) **Using truth table check whether the following statement pattern is tautology, contradiction or contingency.**

$[p \to (q \vee r)] \wedge (\sim p)$

Solution :

p	q	r	$\sim p$	$q \vee r$	$p \to (q \vee r)$	$[p \to (q \vee r)] \wedge (\sim p)$
T	T	T	F	T	T	F
T	T	F	F	T	T	F
T	F	T	F	T	T	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

Since in the last column, the truth values of the statement pattern is neither all T nor all F. Hence , it is contingency.

Q. 10) **Show that the following pair of statements are logically equivalent.**

$(p \leftrightarrow q), (p \to q) \wedge (q \to p)$

Solution :

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	**T**	**T**
T	F	F	T	**F**	**F**
F	T	T	F	**F**	**F**
F	F	T	T	**T**	**T**

The corresponding truth values in last two columns are identical

Hence $(p \leftrightarrow q) \equiv (p \to q) \wedge (q \to p)$

Q. 11) **Write the dual statement of each of the following compound statements.**

1) 13 is prime number and India is a democratic country.

Sol. : 13 is prime number or India is a democratic country.

2) Radha and Krisha can not read urdu.

Sol. : Radha or Krisha can not read urdu.

3) Karina is CR of the class or everybody likes her.

Sol. : Karina is CR of the class and everybody likes her.

4) If Santosh passes in Accountancy, then Kusum passes in Logic.

Sol. : Santosh does not pass in Accountancy but Kusum passes in Logic.

5) A man is rich iff he buys a bungalow in Juhu.

Sol. : A man is not rich but he buys a bungalow in Juhu OR A man is rich but he is not buying a bungalow in Juhu.

6) A question paper is syllabus based though it is difficult.

Sol. : A question paper is syllabus based or it is difficult.

7) $(p \vee q) \wedge r$

Sol. : $(p \wedge q) \vee r$

8) $t \vee (p \wedge q)$

Sol. : $c \wedge (p \vee q)$

9) $(p \wedge t) \vee (c \vee \sim q)$

Sol. : $(p \vee c) \wedge (t \wedge \sim q)$

10) $\sim q \leftrightarrow (p \vee c)$

Sol. : $\sim q \leftrightarrow (p \vee c)$

$\equiv [q \vee (p \vee c)] \wedge [\sim q \vee \sim (p \vee c)]$

$\therefore$ dual of the statement is

$[q \wedge (p \wedge t)] \vee [\sim q \wedge \sim (p \wedge t)]$

Q. 12) Write negation of the following statements.

1) If Swara gets fast train then she will reach college on time.

Sol. : Let p : Swara gets fast train and q : Swara will reach college on time.

$\therefore \sim (p \rightarrow q) \equiv \sim (\sim p \vee q) \equiv p \wedge \sim q$

$\therefore$ negation of the given statement is :

Swara gets fast train but she will not reach college on time .

2) No Blackberry or Nokia mobile are there in the market. $\sim p \wedge \sim q$

Sol. : Some Blackberry or Nokia mobile are there in the market. **OR** at least one Blackberry or Nokia mobile in the market.

3) There are some people who like to help in cleaning the environment.

Sol. : None of the people like to help in cleaning the environment. **OR** All the people don't like to help in cleaning the environment.

4) No one is perfect.

Sol. : There exist at least one who is perfect. OR Some are perfect.

5) A quadrilateral is a rectangle iff its diagonals bisects each other.

Sol. : A quadrilateral is rectangle but its diagonals don't bisect each other or a quadrilateral is not rectangle but its diagonals bisects each other.

6) $\forall$ n $\in N$, n + 7 > 6.

Sol. : $\exists$ n $\in N$, such that n + 7 $\leq$ 6.

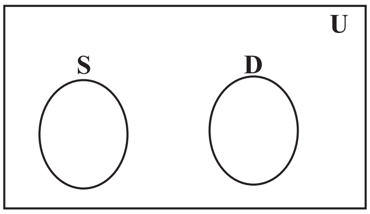
7) $\exists\, x \in$ A, such that $x + 5 > 8$.

Sol. : $\forall\, x \in$ A , $x + 5 \leq 8$.

Q. 13) Represent the truth of the following statement by venn diagrams.

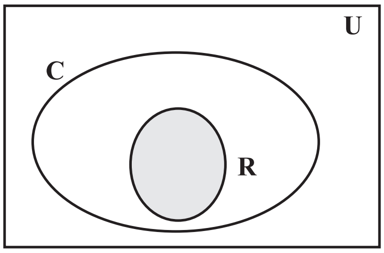
1) No singers are dancers.

Sol. : Let U : Set of all human beings.
S : Set of all singers
D : Set of all dancers.



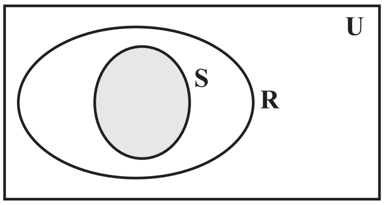
2) All real numbers are complex numbers.

Sol. : Let U : Set of all n-dimensional numbers.
R : Set of all real numbers
C : Set of all complex numbers.



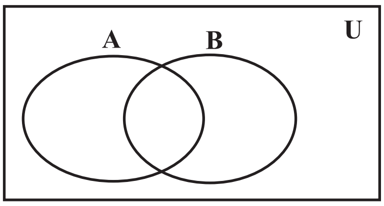
3) Some rectangles are squares.

Sol. : Let U : Set of all geometrical shapes.
R : Set of all rectangles.
S : Set of all squares.



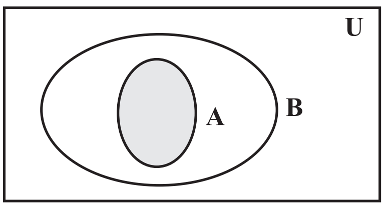
4) Some multiples of 5 are also multiples of 6.

Sol. : Let U : Set of all numbers.
A : Set of all multiple of 5 numbers.
B : Set of all multiple of 6 numbers.



5) If x is prime and $x \neq 2$ then it is odd.

Sol. : Let U : Set of all real numbers.
A : x is prime and $x \neq 2$.
B : Set of all odd numbers.



Question Bank :

Q. 1) Select and write the most appropriate answer from the given alternatives for each sub-question.

1) Which of the following is not a statement?

a) $2 + 3 = 4$ b) 13 is odd number c) go there d) triangle has 3 sides.

2) Which of the following is an open statement ?

a) Wish you all the best b) keep quite c) x is a real number. d) come here.

3) Let $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$, then this law is known as

a) Commutative law b) Associative law

c) D'morgan's law d) Distributive law

4) The statement $(\sim p \wedge q) \vee \sim q$ is ....

a) $p \vee q$ b) $p \wedge q$ c) $\sim (p \vee q)$ d) $\sim (p \wedge q)$

5) If p and q are two statements then $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is

a) Contradiction b) tautology c) contingency d) none of these

6) Conditional $p \rightarrow q$ is equivalent to ....

a) $p \rightarrow \sim q$ b) $\sim p \vee q$ c) $\sim p \rightarrow \sim q$ d) $p \vee \sim q$

7) If p is any statement then $(p \vee \sim p)$ is a ....

a) Contradiction b) tautology c) contingency d) none of these

8) If p : $2 + 3 = 5$, q : sum of interior angles of a triangle is 90 degree.

Then, symbolic form of statement "It is false that, $2 + 3 = 5$ and sum of interior angles of a triangle is 90 degree.

a) $p \vee q$ b) $\sim p \wedge q$ c) $\sim (p \vee q)$ d) $\sim (p \wedge q)$

9) If $p \rightarrow q$ is an implication, then the implication $\sim q \rightarrow \sim p$ is called its

a) Converse b) contra positive c) inverse d) alternative

10) The dual of the statement $(\sim p \wedge q) \vee (\sim q \wedge s)$ is ....

a) $(\sim p \vee q) \wedge (\sim q \vee s)$ b) $(\sim p \rightarrow q) \vee (\sim q \rightarrow s)$

c) $(\sim p \rightarrow q) \wedge (\sim q \rightarrow s)$ d) $(p \wedge \sim q) \vee (q \wedge \sim s)$

11) Negation of the conditional Statement "If it rains, I shall go to school." is

a) It rains and I shall go to school. b) It rains and I shall not go to school.

c) It does not rains and I shall go to school. d) none of these.

12) Which of the following is a contradiction. ....

a) $(p \wedge q) \wedge \sim (p \vee q)$ b) $p \vee (\sim p \wedge q)$ c) $(p \rightarrow q) \rightarrow p$ d) none of these

13) Which of the following is logically equivalent to $\sim (\sim p \rightarrow q)$

a) $p \wedge q$ b) $p \wedge \sim q$ c) $\sim p \wedge q$ d) $\sim p \wedge \sim q$

14) $\sim (p \leftrightarrow q)$ is logically equivalent to ....

a) $\sim p \wedge \sim q$ b) $\sim p \vee \sim q$ c) $(p \wedge \sim q) \vee (\sim p \wedge q)$ d) none of these

15) $\sim (p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to ....

a) $\sim p$ b) p c) q d) $\sim q$

16) The inverse of the proposition $(p \wedge \sim q) \rightarrow r$ is ....

a) $\sim r \rightarrow (\sim p \vee q)$ b. $(\sim p \vee q) \rightarrow \sim r$ c) $r \rightarrow (p \wedge \sim q)$ d) none of these

17) If $p \rightarrow (\sim p \vee q)$ is false, the truth values of p and q are respectively

a) F, T b) F, F c) T, T d) T, F

Q. 2) State whether the following statements are true or false.

1) $p \rightarrow q$ is equivalent to $p \rightarrow \sim q$.
2) $p \leftrightarrow q$ is false when p and q have different truth values.
3) Mathematical identities are true statements.
4) $x^2 = 25$ is a true statement.
5) $(\sim p \vee q) \wedge \sim q$ is a contradiction.
6) Converse of $p \rightarrow \sim q$ is $\sim q \rightarrow p$.
7) $\exists\, n \in N$ such that $n + 7 > 12$
8) The negation of $10 + 20 = 30$ is, it is false that $10 + 20 \neq 30$
9) "His birthday is on 1st may." is not a statement.
10) Dual of $(\sim p \vee q) \wedge (\sim q \rightarrow p)$ is $(p \wedge \sim q) \vee (q \rightarrow \sim p)$
11) Truth value of $\sqrt{3}$ is not an irrational number is F.

Q. 3) Fill in the blanks.

1) If $p \wedge q = F$, $p \rightarrow q = F$, then the truth value of p and q are .... and ....
2) The statement $q \rightarrow p$ is called as the .... of the statement $p \rightarrow q$.
3) If $p \wedge q$ is true then truth value of $\sim p \vee q$ is ....
4) Negation of "some complex numbers are integer." is ....
5) Dual of $(p \vee t) \wedge (\sim q \wedge c)$ is ....

Q. 4) Change each of the following statement in the form if .... then ....

1) I shall come provided I finish my work.
2) Right follow from performing the duties sincerely.
3) $x = 1$ only if $x^2 = x$.
4) The sufficient condition for being rich is to be rational.
5) Getting bonus is necessary condition for me to purchase a car.

Q. 5) Construct truth table for each of the following statement pattern and conclude whether it is tautology, contradiction or contingency.

1) $(p \wedge \sim q) \vee (\sim p \wedge q)$
2) $\sim (p \wedge \sim q) \vee \sim p$
3) $[(p \wedge q) \vee r] \wedge [\sim r \vee (p \wedge q)]$
4) $[(p \wedge q) \wedge (q \rightarrow r)] \leftrightarrow (p \rightarrow r)$

Q. 6) Write converse, inverse and contra positive of the following statements.

1) If an angle is right angle then its measure is 90°.
2) If two triangles are congruent then their areas are equal.
3) If $f(2) = 0$ then $f(x)$ is divisible by $(x - 2)$

Q. 7) Write the negation of following statements.

1) All equilateral triangles are isosceles.
2) Some complex numbers are not real numbers.
3) Every student has paid the fees.
4) $\forall\ n \in N$, $n - 8 > 9$.
5) $\exists\ x \in R$, such that $x^2 > x$
6) Democracy survives if the leaders are not corrupt.
7) The necessary and sufficient condition for a person to be successful is to be honest.

Q. 8) Represent the truth of the following statement by venn diagrams.

1) Some real numbers are not rational numbers.
2) No student is lazy.
3) People are happy iff their concept of maths are clear.
4) Some of the commerce students have not taken maths.
5) Sunday implies holiday.

■■■

2. Matrices

Introduction :

Definition (Matrix) :

A rectangular arrangements of mn numbers in m rows and n columns, enclosed in [ ] or ( ) is called a matrix of order m by n.

Types of Matrices :

1) **Row matrix :** It is a matrix that have only one row.
2) **Column matrix :** It is a matrix that have only one column.
3) **Zero matrix :** It is a matrix whose all elements are zero. It is denoted by letter O.
4) **Square matrix :** It is a matrix in which numbers of rows equal to number of column. Otherwise Matrix is **Rectangular.**

   Note : In square matrix, element of type a_{11}, a_{22}, a_{33}.... are called diagonal elements and other elements a_{ij}, $i \neq j$ are called non- diagonal elements.
5) **Diagonal matrix :** It is a square matrix in which all non-diagonal elements are zero.
6) **Scalar matrix :** It is a square matrix in which (i) all non-diagonal elements are zero. (ii) all diagonal elements are equal.
7) **Unit matrix :** It is a square matrix in which (i) all non-diagonal elements are zero. (ii) all diagonal elements are equal to unity (1). It is denoted by letter *I*.
8) **Upper Triangular matrix :** It is a square matrix in which every element below the diagonal is zero.
9) **Lower Triangular matrix :** It is a square matrix in which every element above the diagonal is zero.
10) **Triangular matrix :** It is a square matrix which is either Upper or Lower Triangular.
11) **Singular matrix :** It is a square matrix whose determinant is equal to zero. Otherwise it is said to be a **Non-singular matrix.**

Examples of Singular matrix	Examples of Non- singular matrix
$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ $\lvert A \rvert = (2 \times 6) - (4 \times 3) = 12 - 12 = 0$	$B = \begin{bmatrix} 4 & 3 \\ 2 & -2 \end{bmatrix}$ $\lvert B \rvert = [4 \times (-2)] - (3 \times 2)$ $= -8 - 6 = 14 \neq 0$

Examples of Singular matrix	Examples of Non- singular matrix
$\|C\| = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$	$\|D\| = \begin{bmatrix} 2 & -1 & 3 \\ -7 & 4 & 5 \\ -2 & 1 & 6 \end{bmatrix}$
$= 2\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix} + 4\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix}$	$= 2\begin{vmatrix} 4 & 5 \\ 1 & 6 \end{vmatrix} + 1\begin{vmatrix} -7 & 5 \\ -2 & 6 \end{vmatrix} + 3\begin{vmatrix} -7 & 4 \\ -2 & 1 \end{vmatrix}$
$= 2\ (24 - 25) - 3\ (18 - 20) + 4\ (15 - 16)$	$= 2\ (24 - 5) + 1\ (- 42 + 10) + 3\ (- 7 + 8)$
$= 2 + 6 - 4 = 0$	$= 38 - 32 + 3 = 9 \neq 0$

Solved Examples :

Q. 1) Construct a matrix$[a_{ij}]_{3 \times 2}$ where element a_{ij}, $i - 3j$.

Solution : $A = [a_{ij}]_{3 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 & 3 \\ -7 & 4 & 5 & 5 \\ -2 & 1 & 6 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ -1 & -4 \\ 0 & 3 \end{bmatrix}$

Q. 2) Find k if the matrix B = $\begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{1} \\ \mathbf{7} & \boldsymbol{k} & \mathbf{1} \\ \mathbf{10} & \mathbf{9} & \mathbf{1} \end{bmatrix}$ **is singular.**

Solution : Matrix B is singular.

$$\therefore |B| = 0 \Rightarrow \begin{bmatrix} 4 & 3 & 1 \\ 7 & k & 1 \\ 10 & 9 & 1 \end{bmatrix} = 0$$

$$\Rightarrow 4\ (I - 9) - 3\ (7 - 10) + 1\ (63 - 10k) = 0$$

$$\Rightarrow 4k - 36 + 9 + 63 - 10k = 0$$

$$\Rightarrow - 6k + 36 = 0 \Rightarrow 6k = 36 \Rightarrow k = 6.$$

Algebra of Matrices :

Addition, Subtraction and Scalar Multiplication :

Q. 3) If $A = \begin{bmatrix} 5 & -3 \\ 1 & 0 \\ -4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 2 & 7 \\ -3 & 1 \\ 2 & -2 \end{bmatrix}$ find $2A - 3B$.

Solution : $2A - 3B = 2\begin{bmatrix} 5 & -3 \\ 1 & 0 \\ -4 & -2 \end{bmatrix} - 3\begin{bmatrix} 2 & 7 \\ -3 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -6 \\ 2 & 0 \\ -8 & -4 \end{bmatrix} - \begin{bmatrix} 6 & 21 \\ -9 & 3 \\ 6 & -6 \end{bmatrix}$

$$= \begin{bmatrix} 10-6 & -6-21 \\ 2+9 & 0-3 \\ -8-2 & -4+6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -27 \\ 11 & -3 \\ -14 & 2 \end{bmatrix}$$

Q. 4) If $A = \begin{bmatrix} 2x+1 & -1 \\ 3 & 4y \end{bmatrix} + \begin{bmatrix} -1 & 6 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$

Solution : Give $\begin{bmatrix} 2x+1 & -1 \\ 3 & 4y \end{bmatrix} + \begin{bmatrix} -1 & 6 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$

$$\begin{bmatrix} 2x+1-1 & -1+6 \\ 3+3 & 4y+6 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 2x & 5 \\ 6 & 4y \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$$

by Equality of two matrices definition,

$2x = 4$ and $4y = 12$

$x = 2$ and $y = 3$

Multiplication of Matrices :

Q. 5) Let $A = [1\ 2\ 3]$ and $B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, find AB and BA.

Solution : $AB = [1\ 2\ 3] \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = [(1 \times 3)] + (3 \times 2)] = [3 + 6 + 2] = [11]$

$$BA = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} [1\ 2\ 3] = \begin{bmatrix} 3 \times 1 & 3 \times 3 & 3 \times 2 \\ 2 \times 1 & 2 \times 3 & 2 \times 2 \\ 1 \times 1 & 1 \times 3 & 1 \times 2 \end{bmatrix}$$

Q. 6) Let A = $\begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$, find AB.

Solution : $AB = \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$

$$= \begin{bmatrix} (4 \times 1) + (-3 \times 4) & (4 \times 3) + (-3 \times -2) \\ (5 \times -1) + (2 \times 4) & (5 \times 3) + (2 \times -2) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -12 & 12 + 6 \\ -5 & +8 & 15 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & 18 \\ 3 & 11 \end{bmatrix}$$

Rules of Matrix :

1) **Transpose of matrix :** Let A is a matrix of order $m \times n$. Matrix obtained by interchanging rows and columns of matrix A is called the transpose of the matrix A. It is denoted by A' or A^T.

   (i) Symmetric matrix : A square matrix $A = [a_{ij}]_{n \times n}$ in which $a_{ij} = a_{ij}$ for all i and j is called a symmetric matrix.

   (ii) Skew Symmetric matrix : A square matrix $A = [a_{ij}]_{n \times n}$ in which $a_{ij} = -\, a_{ij}$ for all i and j is called a symmetric matrix. (All diagonal elements are zero.)

Note : i) If A is Symmetric Matrix then $A = A^T$.

ii) If B is skew symmetric Matrix then $B = -B^T$.

iii) $(A^T)^T = A$

2) **Equality of two matrices :** Two Matrices A and B are said to be equal if (i) Order of A = Order of B (ii) corresponding elements of matrix A and B are same.

Q. 7) If $\begin{bmatrix} 2a - b & 4 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -7 & a + 3b \end{bmatrix}$ then find a and b.

Solution : If $\begin{bmatrix} 2a - b & 4 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -7 & a + 3b \end{bmatrix}$

By equality of matrices definition

$2a - I = 1$ .... (1)

$a + 3b = 2$ .... (2)

Solving (1) and (2). we get

$a = \dfrac{5}{7}$ and b $a = \dfrac{3}{7}$

Inverse of a matrix :

The inverse of a matrix (if it exists) can be obtained by using two methods.

(i) Adjoint method (ii) Elementary row or column transformation

Inverse of a square matrix by adjoint method :

Let us first recall the definition of minor and co-factor of an element of a Matrix.

Definition : Minor of an element a_{ij}, of a matrix is the determinant obtained by deleting i^{th} row and j^{th} column in which the element a_{ij} lies. Minor of an element a_{ij}, is denoted by M_{ij}.

Definition : Co-factor of an element a_{ij}, of a matrix is given by $(-1)^{i+j} M_{ij}$, where M_{ij} is minor of the element a_{ij}. Co-factor of an element a_{ij} is denoted by A_{ij}.

Definition : Co-factor matrix is a matrix obtained by replacing every element of matrix by its cofactor.

Definition : The adjoint of a square matrix $A = [a_{ij}]_{m\times m}$ is defined as the transpose of the co factor matrix .The adjoint of the matrix A is denoted by *adj A*.

Theorem : If $A = [a_{ij}]_{m\times m}$ is a non-singular square matrix then its inverse exists and it is given by $A^{-1} = \frac{1}{|A|}(adjA)$.

Q. 9) Find the adjoint of matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

Solution : Here

$a_{11} = 2$ $\therefore M_{11} = 1$ and $A_{11} = (-1)^{1+1}(1) = 1$

$a_{12} = -3$ $\therefore M_{12} = 4$ and $A_{12} = (-1)^{1+2}(4) = -4$

$a_{21} = 4$ $\therefore M_{21} = -3$ and $A_{21} = (-1)^{2+1}(-3) = 3$

$a_{22} = 1$ $\therefore M_{22} = 2$ and $A_{22} = (-1)^{2+2}(2) = 2$

$\therefore$ Co-factor matri $\begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix}$ $\therefore$ $adjA$ = transpose of co-factor matrix.

$$\therefore adj\ A = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$$

Q. 10) If = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ verify that $A\ (adj\ A) = (adj\ A)\ A = |A|\ I$.

Solution :

$A_{11} = (-1)^{1+1}(4) = 4,$ $A_{12} = (-1)^{1+2}(3) = -3,$

$A_{21} = (-1)^{2+1}(2) = -2,$ $A_{22} = (-1)^{2+2}(1) = 1$

$$adj\ A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore\ A\ (adj\ A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{.... (i)}$$

$$(adj\ A)\ A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4-6 & 8-8 \\ -3+3 & -6+4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{.... (ii)}$$

$$\text{And } |A|\ I = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{.... (iii)}$$

From (i), (ii) and (iii) we get, $A(adjA) = (adj\ A)\ A = |A|\ I$

Q. 11) If A = $\begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$**, find A^{-1} by the adjoint method**.

Solution :

Step 1 : $|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2\ (4-1) + (-2+1) + 1\ (1-2) = 6 - 1 - 1 = 4$

Step 2 : For the given matrix A

$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \qquad A_{12} = (-1)^{1+2} \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = 1 \qquad A_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1$

$A_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} = 1 \qquad A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \qquad A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = 1$

$A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = -1 \qquad A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} = 1 \qquad A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 1$

$$\therefore\ adjA = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Step 3 :

Now Therefore by using the formula for A^{-1}

$\therefore\ A^{-1} = \frac{1}{|A|}\ (adjA) \qquad \therefore\ A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

Elementary Transformations :

Inverse of a non-singular matrix by elementary transformations :

By definition of inverse of A, if A^{-1} exists then $AA^{-1} = A^{-1}A = I$.

Remark : To convert $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ into identity matrix we have to apply following.

Row Transformation :	Column Transformation :
i) Reduce a_{11} into 1 by applying $R_1 \leftrightarrow R_2$ Or $\frac{1}{a_{11}} R_1$.	i) Reduce a_{11} into 1 by applying $C_1 \leftrightarrow C_2$ Or $\frac{1}{a_{11}}$ i_1.
ii) Reduce a_{21} into 0 by applying $R_2 - a_{21}R_1$.	ii) Reduce a_{12} into 0 by applying $C_2 - a_{12}C_1$.
iii) Reduce a_{22} into 1 by applying $\frac{1}{a_{22}} R_2$.	iii) Reduce a_{22} into 1 by applying $\frac{1}{a_{22}} C_2$.
iv) Reduce a_{12} into 0 by applying $R_1 - a_{12} R_2$	iv) Reduce a_{21} into 0 by applying $C_1 - a_{21} C_2$.

Q. 12) Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ using row transformation.

Solution :

Step 1 : $|A| = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -2 \therefore |A| \neq 0 \therefore A^{-1}$ exists.

Step 2 :

AA^{-1} $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\boldsymbol{R_2 - 3R_1}$ $\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ **Using $\frac{-1}{2} R_2$ We get**	$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{-3}{2} & \frac{-1}{2} \end{bmatrix}$ **$R_2 - 3R_1$ We get** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{-3}{2} & \frac{-1}{2} \end{bmatrix}$ $\therefore A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{-3}{2} & \frac{-1}{2} \end{bmatrix}$

Application of Matrices :

Now we intend to discuss the application of matrices for solving a system of linear equations. For this we first learn to convert the given system of equations in the form of a matrix equation i.e $AX = B$. There are two methods for this application which are namely.

1) Method of inversion **2) Method of reduction.**

Questions for practice:

Q. 1) Select and write the correct option from the given alternatives. [1 marks each]

1) If $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then $AB =$ ....

a) $\begin{bmatrix} \frac{3}{5} \\ \frac{3}{7} \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ c) $\begin{bmatrix} \frac{7}{3} \\ \frac{5}{3} \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2) The matrix $\begin{bmatrix} 5 & -2 & 9 \\ -2 & 7 & 8 \\ 9 & 8 & 3 \end{bmatrix}$ is ....

a) Scalar Matrix b) Symmetric Matrix

c) Skew symmetric Matrix d) Diagonal Matrix

3) If A is a 2 × 2 matrix such that $A.\ adj\ A = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$ then $|A| =$ ....

a) 0 b) 7 c) 10 d) 49

4) If $A = \begin{bmatrix} 3 & 6 \\ -2 & 8 \end{bmatrix}$ then cofactor of a_{12} is ....

a) 2 b) 8 c) 3 d) 6

5) Adjoint of $B = \begin{bmatrix} 4 & -3 \\ 6 & 8 \end{bmatrix}$ is ....

a) $\begin{bmatrix} 2 & 4 \\ -3 & 6 \end{bmatrix}$ b) $\begin{bmatrix} 2 & -3 \\ 6 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 4 & 6 \\ -3 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 3 \\ -6 & 4 \end{bmatrix}$

6)) Matrix $B = \begin{bmatrix} 0 & q & 9 \\ -4 & 0 & p \\ -9 & -6 & 0 \end{bmatrix}$ is skew symmetric then the value of p and q are ....

a) 6, 4 b) 6, − 4 c) −6, 4 d) 9, 6

7) If $A = \begin{bmatrix} \alpha & 4 \\ 4 & \alpha \end{bmatrix}$ and $|A^3| = 729$ then the values of α are ....

a) ± 3 b) ± 4 c) ± 5 d) ± 6

8) If $A = \begin{bmatrix} 6 & 1 \\ 11 & 2 \end{bmatrix}$ then A^{-1} = is ....

a) $\begin{bmatrix} 6 & 1 \\ 11 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 \\ 11 & 6 \end{bmatrix}$ c) $\begin{bmatrix} 2 & -1 \\ -11 & 6 \end{bmatrix}$ d) $\begin{bmatrix} -2 & 1 \\ 11 & -6 \end{bmatrix}$

Q. 2) State Whether each of the following is True or False. [1 marks each]

1) If A is non-singular then $|A| = 0$.

2) If A is symmetric then $A = -A^T$.

3) Every diagonal matrix is scalar matrix.

4) If A and B are conformable for the product AB then $(AB)^T = B^T . A^T$

5) $A = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$ is not invertible matrix.

6) $A = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$.

7) If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ then $|adjA|$ is.a^9.

8) $A = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is row matrix.

Q. 3) Fill in the blanks. **[1 marks each]**

1) Co factor matrix of $\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}$ is ....

2) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $[2 \ -4 \ 3]$ = ...

3) If $\begin{bmatrix} 2x+1 & -1 \\ 3 & 4y \end{bmatrix} + \begin{bmatrix} -1 & 10 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 12 & 12 \end{bmatrix}$ is then the value of x and y are ....

4) If $A = \begin{bmatrix} 7 & 3 \\ -2 & k \end{bmatrix}$ is singular then the value of k is ....

5) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}$ then $A + B^T$........... .

6) If $A = \begin{bmatrix} a & b & c \\ p & q & r \\ 2a-p & 2b-q & 2c-r \end{bmatrix}$ then $|A|$ = ...

7) If we interchange R_1 with R_2 in matrix $A = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$ then resulting matrix is ....

8) If $A = [a_{ij}]_{3\times 3}$ Where $a_{ij} = i - j$ then the matrix A = ...

Q. 4) Solve the following. **[3 marks each]**

1) If $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \\ -6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 4 & 2 \\ 1 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ -1 & -4 \\ -3 & 6 \end{bmatrix}$, *Find matrix X, such that*

$3A - 4B + 5X = C$.

2) if $2A - B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$, and $A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$

then $A + B^T$........... .

3) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A$ is a scalar matrix.

4) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ prove that $A^2 - 5A + 7I = 0$ where I is unit matrix of order 2×2.

5) Find the inverse of matrix $\begin{bmatrix} 3 & 1 & 5 \\ 2 & 7 & 8 \\ 1 & 2 & 5 \end{bmatrix}$, by the adjoint method.

6) Find x, y, z if $\left\{ 5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 1 & 3 \end{bmatrix} \right\} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x - 1 \\ y + 1 \\ 2z \end{bmatrix}$

7) Solve the following equations by method of inversion.
$4x - 3y - 2 = 0$, $3x - 4y + 6 = 0$.

8) Express the following equations in matrix form and solve them by method of reduction $3x - y = 1$, $4x + y = 6$.

9) If $A = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}$ then verify $|AB| = |A|\,|B|$.

10) find k, if $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $A^2 = kA - 2I$, where **I** is identity matrix of order 2.

Q. 5) Activity Based Questions : **[4 marks each)**

1) Complete the activity : Solve the following equations by using inversion method.
$x + y - z = 2$, $x - 2y + z = 3$, $2x - y - 3z = -1$.

Solution :

step 1 : Matrix From $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ..... \\ ..... \\ ..... \end{bmatrix}$	**step 2 :** Calculation of A^{-1}.	**step 3 :** Pre multiplying (i) by A^{-1}.
$AX = B$ (i)	1) $\|A\| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & -1 & -3 \end{vmatrix}$ $= 7 + 5 - 3 = 9 \neq 0$ 2) $adjiA = \begin{bmatrix} 7 & 4 & -1 \\ \square & -1 & -2 \\ 3 & 3 & -3 \end{bmatrix}$	$A^{1}(AX) = A^{-1}B$ $IX = A^{-1}B$ $X = A^{-1}B$ $X = \frac{1}{9} \begin{bmatrix} 7 & 4 & -1 \\ 5 & -1 & -2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ $= \square$
	3) $A^{-1} = \frac{1}{\|A\|}$ adj$A$ $= \square$	$x = 3$, $y = 1$, $z = 2$.

1) The sum of three numbers is 6. If we multiply third number by 3 and add it the second number we get 11. By adding first and third number we get a number which is double the second number. Use this information and find a system of linear equations. Find the three numbers by completing the following activity.

Solution : Let the 3 numbers are x , y and z.

$x + y + z \quad 6$

Given $y + 3z = 11$

$\boxed{} = 0$

step 1 : Matrix From

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ldots.. \\ \ldots.. \\ \ldots.. \end{bmatrix}$$

$R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ \ldots.. & \ldots. & \ldots. \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} x + y + z \\ y + 3z \\ -3y \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -6 \end{bmatrix}$$

By equality of two matrices, we get

$x + y + z = 6$ (iv)

$y + 3z = 11.$ (v)

$-3y = -6$ (vi)

$\therefore \ (x, y, z) = \boxed{}$

■■■

3. Differentiation

Rules of Differentiation : If u and v are differentiable functions of x such that

i) $y - u \pm v$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} \pm \dfrac{dy}{dx}$

ii) $y = u.v$ then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

iii) $y = \dfrac{u}{v}$, where $v \neq 0$ then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

- **Derivatives of some standard function :**

$y = f(x)$	$f'(x) = \dfrac{dy}{dx}$
C (Constant)	0
x^n	nx^{n-1}
$\dfrac{1}{x}$	$-\dfrac{1}{x^2}$
$\dfrac{1}{x^n}$	$-\dfrac{n}{x^{n+1}}$
$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$
e^x	e^x
a^x	$a^x \log a$
$\log x$	$\dfrac{1}{x}$
$\log_a x$	$\dfrac{1}{x\log a}$

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x such that the composite function $y = f[g(x)]$ is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

- **Derivatives of some standard Composite Functions :**

$y = f(x)$	$\frac{dy}{dx}$
$[f(x)]^n$	$n[f(x)]^{n-1} f'(x)$
$\sqrt{fx}$	$\frac{1}{2\sqrt{x}} f'(x)$
$\frac{1}{[f(x)]^n}$	$\frac{-n.f'(x)}{[f(x)]^{n+1}}$
a^{fx}	$a^{fx} \log a\, f'(x)$
$e^{f(x)}$	$e^{f(x)} f'(x)$
$\log f(x)$	$\frac{f'(x)}{f(x)}$

Solved Examples :

Q. 1) Differentiate the following w. r. t. x.

(i) $y = \sqrt{x^2+5}$ (ii) $\log \sqrt{x^5+4}$ (iii) $y = \frac{3}{(2x^2-7)^5}$

Sol. : i) $y = \sqrt{x^2+5}$

Method 1 : Let $u = x^2+5$ then $y = \sqrt{u}$, where y is a differentiable function of u and u is a differentiable function of x then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Now, $y = \sqrt{u}$

Differentiate w. r. t. u

$\frac{dy}{du} = \frac{d}{du}(\sqrt{u}) = \frac{1}{2\sqrt{u}}$ and $u = x^2+5$

Differentiate w. r. t. x $\frac{du}{dx} = \frac{d}{dx}(x^2+5) = 2x.$

Now, equation (I) becomes, $\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 2x = \frac{x}{\sqrt{x^2+5}}$

Method 2 : We have y = $\sqrt{x^2+5}$

Differentiate w. r. t. x, $\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x^2+5}\right)$

[Treat x^2+5 as u in mind and use the formula of derivative of $\sqrt{u}$]

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+5}} \times \frac{d}{dx}(x^2+5) = \frac{1}{2\sqrt{x^2+5}} \times 2x = \frac{x}{\sqrt{x^2+5}}$$

ii) Let y = log (x^5+4)

Differentiate w. r. t. x

$$\frac{du}{dx} = \frac{d}{dx}[\log(x^5+4)]$$

$$\frac{du}{dx} = \frac{1}{x^5+4} \times \frac{d}{dx}(x^5+4)$$

$$\frac{dy}{dx} = \frac{1}{x^5+4}(5x^4) = \frac{5x^4}{x^5+4}$$

iii) Let y = $\frac{3}{(2x^2-7)^5}$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} = \left(\frac{3}{(2x^2-7)^5}\right) = 3\frac{d}{dx}\left(\frac{1}{(2x^2-7)^5}\right)$$

$$= 3 \times \frac{-5}{(2x^2-7)^6} \times \frac{d}{dx}(2x^2-7)$$

$$= -\frac{15}{(2x^2-7)^6}(4x)$$

$$\frac{dy}{dx} = -\frac{60x}{(2x^2-7)^6}$$

Derivatives of Implicit Functions :

1) Differentiate both sides of the equation with respect to x (independent variable), treating y as a differentiable function of x.

2) Collect the terms containing $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$

Ex. 1) Find $\frac{dy}{dx}$ if

i) $x^5 + xy^3 + x^2y + y^4 = 4$ ii) $x^5 + e^{xy} = y^2 + \log(x + y)$

Sol. :

i) Given that : $x^5 + xy^3 + x^2y + y^4 = 4$

Differentiate w. r. t. x ,

$$\frac{d}{dx}(x^5) + \frac{d}{dx}(xy^3) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(y^4) = \frac{d}{dx}4$$

$$5x^4 + x\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(x) + x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) + 4y^3\frac{d}{dx}(y) = 0$$

$$5x^4 + x(3y^3)\frac{dy}{dx} + y^3(1) + x^2\frac{dy}{dx} + y(2x) + 4y^3\frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 3xy^2\frac{dy}{dx} + 4y^3\frac{dy}{dx} = -5x^4 - 2xy - y^3$$

$$(x^2 + 3xy^3 + 4y^3)\frac{dy}{dx} = -(5x^4 + 2xy + y^3)$$

$$\therefore \frac{dy}{dx} = -\frac{5x^4 + 2xy + y^3}{x^2 + 3xy^3 + 4y^3}$$

ii) Given that : $x^2 + e^{xy} = y^2 + \log(x + y)$

Differentiate w. r. t. x $\frac{d}{dx}(x^2) + \frac{d}{dx}[e^{xy}] = \frac{d}{dx}(y^2) + \frac{d}{dx}[\log(x + y)]$

$$2x + e^{xy}\frac{dy}{dx}(xy) = 2y\frac{dy}{dx} + \frac{1}{x+y}\frac{d}{dx}(x + y)$$

$$2x + e^{xy}\left[\left[x\frac{dy}{dx} + y(1)\right]\right] = 2y\frac{dy}{dx} + \frac{1}{x+y}\left[1 + \frac{dy}{dx}\right]$$

$$2x + xe^{xy}\frac{dy}{dx} + ye^{xy} = 2y\frac{dy}{dx} + \frac{1}{x+y} + \frac{1}{x+y}\frac{dy}{dx}$$

$$2x + ye^{xy} - \frac{1}{x+y} = 2y\frac{dy}{dx} - xe^{xy}\frac{dy}{dx} + \frac{1}{x+y}\frac{dy}{dx}$$

$$2x + ye^{xy} - \frac{1}{x+y} = \left[2y - xe^{xy} + \frac{1}{x+y}\right]\frac{dy}{dx}$$

$$\frac{2x(x+y)+ye^{xy}(x+y)-1}{x+y} = \left[\frac{2y(x+y)-xe^{xy}(x+y)+1}{x+y}\right]\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2x(x+y)+ye^{xy}(x+y)-1}{2y(x+y)-xe^{xy}(x+y)+1}$$

Ex. 2) Find $x^m.y^n = (x+y)^{m+n}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$.

Sol. :

i) Given that : $x^m.\ y^n = (x+y)^{m+n}$

Taking log on both the sides, we get

$$\log [x^m.\ y^n] = \log [(x+y)^{m+n}]$$

$$m \log x + n\log y = (m + n) \log (x+y)$$

Differentiate w. r. t. x

$$m \frac{d}{dx}(\log x) + n \frac{d}{dx}(\log y) = (m + n) \frac{d}{dx}[\log (x+y)]$$

$$\frac{m}{x} + \frac{n}{y} \cdot \frac{dy}{dx} = \frac{m+n}{x+y} \cdot \frac{d}{dx}(x+y)$$

$$\frac{m}{x} + \frac{n}{y} \cdot \frac{dy}{dx} = \frac{m+n}{x+y} \cdot \left[1+\frac{dy}{dx}\right]$$

$$\frac{m}{x} + \frac{n}{y} \cdot \frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y} \cdot \frac{dy}{dx}$$

$$\frac{n}{y} \cdot \frac{dy}{dx} - \frac{m+n}{x+y} \cdot \frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\left[\frac{n}{y} - \frac{m+n}{x+y}\right]\frac{dy}{dx} = \frac{m+n}{x+y} \quad \frac{m}{x}$$

$$\left[\frac{n(x+y) - (m+n)y}{y(x+y)}\right]\frac{dy}{dx} = \left[\frac{(m+n)x - m(x+y)}{x(x+y)}\right]$$

$$\left[\frac{nx + ny - my - ny)}{x(x+y)}\right]\frac{dy}{dx} = \left[\frac{mx + nx - mx - my}{x}\right]$$

$$\left[\frac{nx - my}{y}\right]\frac{dy}{dx} = \left[\frac{nx - my}{x}\right] \therefore \frac{dy}{dx} = \frac{y}{x}$$

Derivatives of Parametric Functions :

If $x = f(t)$ and $y = g(t)$ are differentiable functions of t so that y is a differentiable function of x and if $\frac{dx}{dt} \neq 0$ then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Ex. 1) Find $\frac{dy}{dx}$ if i) $x = at^4$, $y = 2at^2$ ii) $x = t - \sqrt{t}$, $y = t + \sqrt{t}$

Sol. : i) $x = at^4$, $y = 2at^2$

Given $y = 2at^2$

Differentiate with respect to t, we get

$$\frac{dy}{dt} = 4at$$

And $x = at^4$

Differentiate with respect to t, we get

$$\frac{dx}{dt} = 4at^3$$

Now $\frac{dy}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at}{4at^3} = \frac{1}{t^2}$

ii) $x = t - \sqrt{t}$, $y = t + \sqrt{t}$

Given $y = t + \sqrt{t}$

Differentiate w. r. to t

$$\frac{dy}{dt} = 1 + \frac{1}{2\sqrt{t}} = \frac{2\sqrt{t} + 1}{2\sqrt{t}}$$

And $x = t - \sqrt{t}$

Differentiate w. r. to t

$$\frac{dx}{dt} = 1 - \frac{1}{2\sqrt{t}} = \frac{2\sqrt{t}-1}{2\sqrt{t}}$$

$$\text{Now } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} - \frac{\frac{(2\sqrt{t}+1)}{2\sqrt{t}}}{\frac{(2\sqrt{t}-1)}{2\sqrt{t}}} = \frac{2\sqrt{t}+1}{2\sqrt{t}-1}$$

Ex. 2) If $ax^2 + 2hxy + by^2 = 0$ then show that $\frac{d^2y}{dx^2} = 0.$

Sol. : Given that $ax^2 + 2hxy + by^2 = 0$ ... (I)

$$ax^2 + cxy + hxy + by^2 = 0$$

$$x(ax + hy) + y(hx + by) = 0$$

$$y(hx + by) = -x(ax + hy)$$

$$\frac{y}{x} = -\frac{ax+hy}{hx+by} \quad \text{... (II)}$$

Differentiate (I) w. r. t. x

$$a\frac{d}{dx}(x^2) + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(y^2) = 0$$

$$a(2x) + 2h\left[x\frac{dy}{dx} + y(1)\right] + b(2y)\frac{dy}{dx} = 0$$

$$2\left[ax + 2x\frac{dy}{dx} + hy + by\frac{dy}{dx}\right] = 0$$

$$(hx + by)\frac{dy}{dx} = -ax - hy$$

$$\frac{dy}{dx} = -\frac{ax+hy}{hx+by}$$

From (II), we get

$$\therefore \frac{dy}{dx} = \frac{y}{x} \quad \text{... (III)}$$

Differentiate (III), w. r. t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx} - y(1)}{x^2} = \frac{x\left(\frac{y}{x}\right) - y}{x^2} \quad \text{... [From (II)]}$$

$$\frac{d^2y}{dx^2} = \frac{y - y}{x^2} = 0$$

PROBLEMS FOR PRACTICE

I) Choose the correct alternatives :

1) If $y = \frac{1}{\sqrt{3x^2 - 2x - 1}}$ then $\frac{dy}{dx} = ?$

a) $\frac{-2}{3}(3x - 2)(3x^2 - 2x - 1)^{\frac{-2}{3}}$ b) $\frac{-3}{2}(3x - 2)(3x^2 - 2x - 1)^{\frac{-3}{2}}$

c) $(3x - 1)(3x^2 - 2x - 1)^{\frac{-2}{3}}$ d) $-(3x - 1)(3x^2 - 2x - 1)^{\frac{-3}{2}}$

2) If $y = \sqrt[3]{(3x^2+8x-6)^5}$ then $\frac{dy}{dx} = ?$

a) $\frac{5}{3}(6x + 8)(3x^2+8x-6)^{\frac{2}{3}}$ b) $\frac{-5}{3}(6x + 8)(3x^2 + 8x - 6)^{\frac{2}{3}}$

c) $\frac{3}{5}(6x + 4)(3x^2+8x-6)^{\frac{2}{3}}$ d) $\frac{-3}{5}(3x + 4)(3x^2 + 8x - 6)^{\frac{2}{3}}$

3) What is the rate of change of demand (x) of a commodity with respect to it's price (y) if $y = 10 + x + 25x^3$

a) $\frac{10}{1+75x^2}$ b) $\frac{1}{1+75x^2}$ c) $1+75x^2$ d) $\frac{-1}{1+75x^2}$

4) What is the rate of change of demand (x) of a commodity with respect to it's price (y) if $y = \frac{3x+7}{2x^2+5}$

a) $\dfrac{(2x^2+5)^2}{(-6x^2-38x+15)}$ b) $\dfrac{(2x^2+5)^2}{(-6x^2-28x+15)}$ c) $\dfrac{(2x^2+5)^2}{(6x^2-28x+15)}$ d) $\dfrac{(2x^2+5)^2}{(6x^2-38x+15)}$

5) If $y = x^{\sqrt{x}}$ then $\dfrac{dy}{dx} = ?$

a) $\dfrac{y}{2\sqrt{x}}(\log x + 2)$ b) $\dfrac{y}{\sqrt{x}}(\log x+2)$ c) $\dfrac{y}{2\sqrt{x}}(\log x-2)$ d) $\dfrac{y}{\sqrt{x}}(\log x-2)$

6) If $y = (x)^x + (10)^x$ then $\dfrac{dy}{dx} = ?$

a) $(x)^x(1-\log x) + 10^x \log 10$ b) $x^x(1 + \log x) - 10^x \log 10$

c) $x(1 + \log x) + 10^x \log 10$ d) $x^x(1 + \log x) + 10^x \log 10$

7) If $x^m . y^n = (x + y)^{(m+n)}$ then $\dfrac{dy}{dx} = ?$

a) $\dfrac{y}{x}$ b) $\dfrac{-y}{x}$ c) $\dfrac{y}{x}$ d) $\dfrac{-y}{x}$

8) If $x^y = 2^{(x-y)}$ then $\dfrac{dy}{dx} = ?$

a) $\dfrac{x\log 2 - y}{x\log 2x}$ b) $\dfrac{x\log 2 + y}{x\log 2x}$ c) $\dfrac{x\log 2 + x}{y\log 2x}$ d) $\dfrac{y\log 2 - x}{x\log 2x}$

9) If $x = 2am$, $y = 2am^2$ where m be the parameter then $\dfrac{dy}{dx} = ?$

a) 2m b) −2m c) −am d) am

10) If $x = a\left(t - \dfrac{1}{t}\right)$, $y = a\left(t + \dfrac{1}{t}\right)$, where t be the parameter then $\dfrac{dy}{dx} = ?$

a) $\dfrac{x}{y}$ b) $\dfrac{-x}{y}$ c) $\dfrac{y}{x}$ d) $\dfrac{-y}{x}$

II) Fill in the blanks :

1) If $y = (5x^3 - 4x^2 - 8x)^9$ then $\dfrac{dy}{dx}$ is............

2) If $y = a^{(1 + \log x)}$ then $\frac{dy}{dx}$ is............

3) The rate of change of demand (x) of a commodity with respect to it's price (y) is.............. if $y = 5 + x^2e^{-x} + 2x$

4) The rate of change of demand (x) of a commodity with respect to it's price (y) is................ if $y = xe^{-x} + 7$

5) If $y = x^{10}$ then $\frac{dy}{dx}$ is............

6) If $y = (e)^{(2x+5)}$ then $\frac{dy}{dx}$ is............

7) If $\sqrt{x} + \sqrt{y} = \sqrt{a}$ then $\frac{dy}{dx}$ is............

8) Differetiation of 5^x w.r.t. $\log x$ then $\frac{dy}{dx}$ is............

9) Differetiation of e^x w.r.t. $\log_e x$ then $\frac{dy}{dx}$ is............

10) If $y = x^2$ then $\frac{d^2y}{dx^2}$ is............

III) State whether each of the following is True or False :

1) If $y = \log(\log x)$ then $\frac{dy}{dx} = \log x$

2) If $y = 10^x + 1$ then $\frac{dy}{dx} = 10^x.\log 10$

3) If $y = x^2$ then the rate of change of demand (x) of a commodity with respect to it's price (y) is $\frac{1}{2x}$

4) If $y = 7x + 1$ then the rate of change of demand (x) of a commodity with respect to it's price (y) is 7

5) If $y = e^x$ then $\frac{dy}{dx} = e^x$

6) If $y = 4^x$ then $\frac{dy}{dx} = 4^x$

7) If $\sqrt{x} + \sqrt{y} = \sqrt{a}$ then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}$

8) If $x^2 + y^2 = a^2$ then $\frac{dy}{dx} = 2x + 2y$

9) If $x = 2at$, $y = 2a$, where t is parameter, then $\frac{dy}{dx} = \frac{1}{t}$

10) If $x = 5m$, $y = m$, where m is parameter, then $\frac{dy}{dx} = \frac{1}{5}$

11) If $y = e^x$ then $\frac{(d^2y)}{(dx^2)} = e^x$

Q. 5) Solve the following :

1) Find $\frac{dy}{dx}$, if $y = (log\ x)^x$

2) Find $\frac{dy}{dx}$, if $y = \sqrt[5]{(3x^2 + 8x + 5)^4}$

3) Find $\frac{dy}{dx}$, if $y = (7x-1)^x$

4) Find rate of change of demand (x) of a commodity with respect to it's price (y) if $y = \frac{3x+7}{2x^2+5}$

5) If $x^5.y^7 = (x + y)^{12}$ then show that, $\frac{dy}{dx} = \frac{y}{x}$

6) If $x^a.y^b = (x + y)^{(a + b)}$ then Show that $\frac{dy}{dx} = \frac{y}{x}$

7) If $x^2 + 6xy + y^2 = 10$ then show that $\frac{d^2y}{dx^2} = \frac{80}{(3x+y)^3}$

Q. 5) Activity

1) $y = (6x^4 - 5x^3 + 2x + 3)^6$ find $\frac{dy}{dx}$

Solution : Given

$y = (6x^4 - 5x^3 + 2x + 3)^6$

Let $u = [6x^4 - 5x^3 + 2x + 3]$

$\therefore y = \boxed{}$

$\therefore \frac{dy}{du} = \boxed{}$

And $\frac{du}{dx} = \boxed{}$

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = \boxed{}$$

2) The rate of change of demand (x) of a commodity with respect to its price (y), if $y = 20 + 15x + x^3$

Solution : Let $y = 20 + 15x + x^3$

Diff. w.r.to x, we get

$$\therefore \frac{dy}{dx} = \boxed{} + \boxed{} + \boxed{}$$

$$\therefore \frac{dy}{dx} = 15 + 3x^2$$

$\therefore$ By derivation of the inverse function

$$\frac{dx}{dy} = \frac{1}{\boxed{}}, \ \frac{dy}{dx} \neq 0$$

$\therefore$ Rate of change of demand with respect to price $= \dfrac{1}{\boxed{} + \boxed{}}$

3) Find $\dfrac{dy}{dx}$, if $y = x^{(x)} + 20^{(x)}$

Solution : let $y = x^{(x)} + 20^{(x)}$

Let $u = x^{\boxed{}}$ and $v = \boxed{}^{x}$

$\therefore y = u + v$

Diff. w.r.to x, we get

$$\therefore \frac{dy}{dx} = \frac{\boxed{}}{dx} + \frac{dv}{\boxed{}} \quad \text{................. (I)}$$

Now, u $= x^x$

Taking log on both sides, we get

$\therefore \log u = x \times \log x$

Diff. w.r.to x

$$\therefore \frac{1}{u}\frac{du}{dx} = x \times \frac{1}{\boxed{}} + \log x \times \boxed{}$$

$$\therefore \frac{dy}{dx} = \underline{u}\ [1 + \log x]$$

$$\therefore \frac{du}{dx}\ x^x\ [1 + \boxed{}\ ] \quad \text{................ (II)}$$

Now, $v = 20^x$

Diff.w.r.to x, we get

$\therefore \dfrac{dv}{dt} = 20^x . \boxed{}$ .... (III)

Substituting equation (II) & (III) in equation (I), we get

$\therefore \dfrac{dy}{dx} = x^x\ [1 + \log x] + 20^x. \log (20)$

4) Find $\dfrac{dy}{dx}$, if $x = e^m$, $y = e^{\sqrt{m}}$

Solution : given, $x = e^m$ and $y = e^{\sqrt{m}}$

Now, $y = e^{\sqrt{m}}$

Diff.w.r.to m

$\therefore \dfrac{dy}{dm} = e^{\sqrt{m}}\ \dfrac{d\boxed{}}{dm}$

$\therefore \dfrac{dy}{dm} = e^{\sqrt{m}} . \dfrac{1}{2\sqrt{m}}$ .... (I)

Now, $x = e^m$

Diff.w. r. to m

$\therefore \dfrac{dx}{dm} = \boxed{}$ .... (II)

Now, $\dfrac{dy}{dx} = \dfrac{\frac{dx}{dm}}{\boxed{}}$

$\therefore\ = \dfrac{e^{\sqrt{m}}}{\dfrac{\boxed{}}{e^m}}$

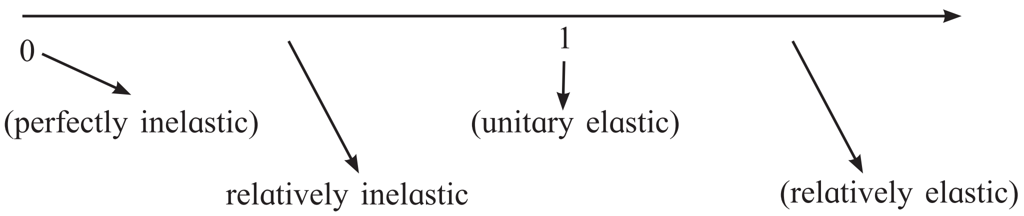
$\therefore \dfrac{dy}{dx} = \dfrac{e^{\sqrt{m}}}{2\sqrt{m}.e^m}$

■■■

4. Application of Derivatives

Formulae/rules/Points to remember :

1) Slope of tangent to curve $y = f(x)$ is $\dfrac{dy}{dx}$ or $f'(x)$
2) Equation of tangent to curve $y = f(x)$ at given point (a, b) is
3) $y - b = f'(a, b)(x - a)$
4) Slope of normal to curve $y = f(x)$ is $-\dfrac{1}{f'(x)}$ and its equation at point (a, b) is $y - b$ $-1\, f'(a, b) = (x - a)$
5) Function $y = f(x)$ is… i) increasing at point $x = a$ if $f'(a) > 0$ and ii) decreasing at point $x = a$ if $f'(a) < 0$
6) Function $y = f(x)$ has maximum at $x = a$ if $f'(a) = 0$ and $f''(a) < 0$ {that means 1st order derivative is 0 ans 2nd order derivative is negative}
7) Function $y = f(x)$ has minimum at $x = a$ if $f'(a) = 0$ and $f''(a) > 0$ {that means 1st order derivative is 0 and 2nd order derivative is positive}
8) 'marginal' means derivative. For eg marginal demand means derivative of demand, marginal profit means derivative of profit.
9) Demand D of a commodity is function of its price p then Elasticity of Demand w.r.t price is $\eta = \dfrac{-p}{D} \quad \dfrac{dD}{dp}$
10) Type of elasticity is decided depending on its value as follows :



11) MR = AR $\left(1 - \dfrac{1}{\eta}\right)$
12) Consumption expenditure E_c is function of income x.

    Marginal Propensity to Consume (MPC) = $\dfrac{d\,Ec}{dx}$

    Average Propensity to Consume (APC) = $\dfrac{E_c}{x}$

    MPS = 1 − MPC, APS = 1 − APC

Solved Examples :

Ex. 1) Find the equation of tangent and normal to the curve $y = 3x^2 + 4x - 5$ at point (1, –3).

Solution : Given curve $y = 3x^2 + 4x - 5$ and Point $P = (1, -3)$

Diff w.r.t.x

$\frac{dy}{dx} = 6x + 4$

i) $\therefore$ slope of tangent at P is $\frac{dy}{dx}$ at $(1, -3) = 6\ (1) + 4 = 10$

$\therefore$ equation of tangent is $y - (-3) = 10\ (x - 1)$

$y + 3 = 10x - 10$

$10x - y = 13$ is the equation of tangent.

ii) slope of normal at P is $\frac{-1}{10}$

$\therefore$ equation of normal is $y - (-3) = \frac{-1}{10}\ (x - 1)$

$10y + 30 = -x + 1$

$\underline{x + 10y = -29}$ is the equation of normal.

Ex. 2) For what values of x the function $f(x) = x^2 + 2x + 7$ increasing.

Solution : $f'(x) = 2x + 2$

function $f(x)$ is increasing, so $f'(x) > 0$

$\therefore\ 2x + 2 > 0$

$\therefore\ 2x > -2$

$\therefore\ x > -1$

i.e $x \in (-1, \infty)$

f is increasing if $x \in (-1, \infty)$

Ex. 3) For what values of x the function $f(x) = x^4 - 3x^3 + 5$ decreasing.

Solution : $f'(x) = 4x^3 + 9x^2$

function $f(x)$ is decreasing, so $f'(x) < 0$

$\therefore\ 4x^3 - 9x^2 < 0$

$\therefore\ x^2(4x - 9) < 0$ (but $x^2 > 0$)

$\therefore\ 4x - 9 < 0$

$\therefore\ x < \frac{9}{4}$

i.e $x \in (-\infty, \frac{9}{4})$ $\therefore$ f is decreasing if $x \in (-\infty, \frac{9}{4})$

Ex. 4) The manufacturing cost of x items is $C = 110 + 2x$ and price p is given by $p = 200 - x$. For what values of x, profit will increase.

Solution : (Here, we have to first get the profit function)

Revenue $R = px = (200 - x)\, x = 200x - x^2$

Cost $C = 110 + 2x$

Profit $P = R - C = (200x - x^2) - (110 + x)$

$= -x^2 + 119x - 110$

p' $= -2x + 119$

Profit is increasing $\therefore$ P' > 0

$\therefore\ -2x + 119 > 0$

$\therefore\ -2x > -119$

$\therefore\ 2x < 119$

$\therefore\ x < 59.5$

$\therefore$ Profit is increasing for $x \in (0, 59.5)$

Ex. 5) The total cost of manufacturing x articles is $C = 59x + 1200\, x^2 - x^4$. Find x for which average cost is i) increasing ii) decreasing.

Solution : Here we have to first find the average cost function.

C_A or $AC = \dfrac{C}{x} = 59 + 1200x - x^3$

Now $C'_A = 1200 - 3x^2$

i) Avg cost is increasing, so $C'_A > 0$

$\therefore\ 1200 - 3x^2 > 0$

$\therefore\ 1200 > 3x^2$

$\therefore\ 3x^2 < 1200$

$\therefore\ x^2 < 400$

$\therefore\ x < 20$

$\therefore$ Avg cost is increasing if $0 < x < 20$

ii) Avg cost is decreasing, so $C'_A < 0$

$\therefore\ 1200 - 3x^2 < 0$

$\therefore\ 3x > 1200$

$\therefore\ x > 20$

$\therefore$ Avg cost is decreasing if $x > 20$

Ex. 6) The total cost of producing x units is ₹ $(x^2 + 60x + 50)$ and the price is ₹ $(180 - x)$ per unit. For how many units the profit will be maximum?

Solution : Here we have to first find the profit function.

Cost $C = x^2 + 60x + 50$

Revenue $R = p.x = x\ (180 - x) = 180x - x^2$

So, Profit $P = R - C$

$= (180x - x^2) - (x^2 + 60x + 50)$

$= -2x^2 + 120x - 50$

Now, $P' = -4x + 120$

$P'' = -4$

Profit is maximum, so $P' = 0$

Therefore, $-4x + 120 = 0$

$x = 30$

and at $x = 30$, $P'' = -4 < 0$

Therefore, $P'' < 0$

Hence, Profit is maximum if $x = 30$

i.e. for 30 units, profit is maximum.

Ex. 7) Divide 70 into two parts so that their product is maximum.

Solution : Let the two parts be x and $(70 - x)$

Let $f(x) = x\ (70 - x) = 70x - x^2$

$f'(x) = 70 - 2x$

$f''(x) = -2$

Here, $f(x)$ is maximum, so $f'(x) = 0$

$70 - 2x = 0$

$x = 35$

and at $x = 35$, $f''(x) = -2 < 0$

Therefore, $f''(x) < 0$

Hence, $f(x)$ is max when $x = 35$

i.e. the required parts are 35 and 35.

Ex. 8) For a commodity, demand function is given by D = 2000 − 10p − p^2. Find elasticity of demand when price (p) is 10 units.

Solution : $D = 2000 - 10p - p^2$

$$\frac{dD}{dP} = -10 - 2p$$

$$\therefore \eta = -\frac{p}{D} \times \frac{dD}{dp}$$

$$= \frac{-p(-10-2p)}{2000-10p-p^2}$$

$$= \frac{p(10+2p)}{2000-10p-p^2}$$

When $p = 10$

$$\eta = \frac{10(10+20)}{2000-100-100}$$

$$= \frac{300}{1800}$$

$$= \frac{1}{6}$$

[Note : as $\eta < 1$, demand is relatively inelastic]

Objective Questions

A) True or false.

State whether the following statements are True or False.

1) A function has minimum at $x = a$, if $f'(a) = 0$ and $f''(a) < 0$
2) $f(x) = 1 - \frac{1}{x}$ is increasing for all x in R ($x \neq 0$)
3) MPC − MPS = 1
4) Function $f(x) = x^2 + 4x - 7$ is increasing if $x < -2$
5) $MR = AR\left(1 - \frac{1}{\eta}\right)$

B) Fill in the blanks.

1) If $f(x) = x^3 + 3x^2 - 3x + 100$, then $f''(x)$ = ...
2) If $0 < \eta < 1$, then the demand is ....
3) If the total cost function $C = x^3 + 5x^2 - 1$, where x is the number of items, then marginal cost is ....

4) The slope of normal to curve $y = 3x^2 - x + 4$ at (1, 3) is ....

5) If marginal propensity to consume for an expenditure function is MPC = 0.7521 then its marginal propensity to save (MPS) is ....

Multiple Choice Questions

C) Select and amp; and write the correct answer from the given alternatives.

1) The slope of tangent to the curve $2x^2 + 3y^2 - 5 = 0$ at (1,1) is :

a) $\frac{2}{3}$ b) $\frac{-2}{3}$ c) $\frac{3}{2}$ d) 1

2) The values of x for which $f(x) = 2x^3 - 15x^2 + 36x + 5$ is decreasing are :

a) $x < 2$ or $x > 3$ b) $2 < x < 3$

c) $x < 2$ d) $x > 3$

3) Function $f(x) = x^2 - 3x + 4$ has minimum value at $x =$

a) $\frac{3}{2}$ b) 0 c) $\frac{-2}{3}$ d) 1

4) If the demand function is $D = 50 - 3p - p^2$ (p is price), elasticity of demand at $p = 5$ is :

a) -6 b) $\frac{5}{9}$ c) 6.5 d) 1

5) If average revenue is 45 and elasticity of demand is 9 then marginal revenue is :

a) 36 b) 40 c) 5 d) 1

D) Solve The following.

1) Find the equations of tangent and normal to the curve $x^2 + y^2 + xy = 3$ at point (1, 1)

2) Test whether the function $f(x) = \frac{5}{x} - 9$ is increasing or decreasing.

3) Find the values of x such that $f(x)$ is decreasing function $f(x) = x^3 - 6x^2 - 15x + 12$.

4) Divide the number 20 into two parts such that their product is maximum.

5) The total cost of producing D units is Rs. $(D^2 + 60D + 70)$ and price is Rs. $(180 - D)$. For what units is the profit maximum?

6) For producing x units of a product, revenue function is $R(x) = 40x - x^2$ and cost function $C(x) = 30 + 4x^2$. Find x so that i) revenue is maximum ii) cost is minimum.

7) For manufacturing x units labour cost is $240 - 54x$ and processing cost is x^2.

Price of each unit is $p = 4800 - 4x^2$. Find the values of x for which

i) total cost is decreasing, ii) revenue is increasing

8) If the demand function is $D = \dfrac{p+6}{p-3}$, find the elasticity of demand at $p = 4$.

9) Find the marginal revenue if the average revenue is 30 and the elasticity of demand is 2.

10) Find MPC, MPS, APC, APS if the expenditure E_c of a person with income I is given as $E_c = (0.0002)\ I^2 + (0.085)\ I$ when $I = 1000$

Activity Based Questions

1) Find the values of x for which $f(x) = x + \dfrac{1}{x}$ is decreasing.

Solution : $f(x) = x + \dfrac{1}{x}$

$$f'(x) = \frac{d}{dx}\left(x + \frac{1}{x}\right) = 1 - \frac{1}{\boxed{}}$$

f is decreasing if $f'(x) < 0$

$$\text{ie if } 1 - \frac{1}{\boxed{}} < 0$$

$$\text{ie if } \boxed{} < 1$$

$$\text{ie if } -1 < \boxed{} < 1$$

so f is decreasing for $x \in (\boxed{})$

2) Complete the following activity to find MPC, MPS, APC, APS if the expenditure E_c of a person with income x is given as $E_c = (0.0003)\ x^2 + (0.075)\ x$ when $x = 1000$

Solution : Given $E_c = (0.0003)\ x^2 + (0.075)\ x$

We have APC $= \dfrac{E_c}{x}$

At $x = 1000$, APC $= \boxed{}$

Also MPC $= \dfrac{d}{dx}\ (E_c)$

At $x = 1000$, MPC $= \boxed{}$

At $x = 1000$, APS $= \boxed{}$

At x =1000 MPS $= \boxed{}$

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5. Integration

Definition :

If $f(x)$ and $g(x)$ are two functions such that $\frac{d}{dx}[g(x)] = f(x)$ then $g(x)$ is called an integral of $f(x)$ with respect to x.

And it is denoted as $\int f(x)dx = g(x)+c$

Where 'c' is called integration constant.

For example : $\frac{d}{dx}(x^4) = 4x^3 \quad \therefore \int 4x^3\, dx = x^4 + c$

Rules of Integration :

If f and g are two real valued integrable function of x and K is constant, then

1) $\int k.f(x)dx = k\int f(x)\, dx$
2) $\int [f(x) + g(x)]dx = \int f(x)\, dx + \int g(x)\, dx$
3) $\int [\, f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

Integration of Some Standard Functions :

Result : If $\int f(x)dx = g(x) + c$ then

$$\int f(ax + b)\, dx = \frac{1}{a} g\,(ax + b) + c$$

1) $$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int f(ax + b)\, dx = \frac{1}{a}\frac{(ax + b)^{n+1}}{n+1} + c$$

2) $$\int \frac{1}{x}dx = \log |x| + c$$

$$\int \frac{1}{(ax + b)}\, dx = \frac{1}{a}\log |\, ax + b| + c$$

3) $$\int a^x\, dx = a^x \log\ a + c$$

$$\int a^{px+q}\, dx = \frac{1}{p}\, a^{px+q} \log a + c$$

4) $$\int e^x\, dx = e^x + c \ \therefore \int e^{ax+b}\, dx = \frac{1}{a}\, e^{ax+b} + c$$

For example :

1) $\int (5x+3)^4\, dx$

$$= \frac{1}{5}\frac{(5\ x + 3)^5}{5} + c$$

$$= \frac{(5\ x + 3)^5}{25} + c$$

2) $\int \frac{1}{(2x+3)} \, dx$

$= \frac{1}{2} \log |2x + 3| + c$

3) $\int 5^{3x + 4} \, dx$

$= \frac{1}{3} 5^{3x + 4} \log 5 + c$

1) $\int e^{7x+5} \, dx$

$= \frac{1}{7} e^{7x+5} + c$

Types of Integration :

Type no. 1 - Method of Substitution : In this method, we reduce the given function to standard form by changing variable x to t, using some suitable substitution $x = \varnothing(t)$.

Result : If $x = \varnothing(t)$. Is a differentiable function of t then.

$\int f(x) \, dx = \int f(\varnothing(t)) \, (\varnothing'(t)dt)$

Corollary 1) If integrand contains function and its derivative then put function equal to t.

i.e. $\int \frac{(\log x)^3}{x} \, dx$

Now, put $\log x = t$

$\frac{1}{x} \, dx = dt$

$\therefore \int t^3 \, dt = \frac{t^4}{4} + c$

$= \frac{(\log x)^3}{4} + c$

Corollary 2) If numerator is perfect derivative of denominator, then integration is *log* of denominator i.e : $\int \frac{f'(x)}{f'(x)} \, dx = \log \, | f(x) | + c$

e.g. $\int \frac{3e^x}{3e^x + 12} \, dx$

Consider $\frac{d}{dx} 3e^x + 12 = 3e^x$

i.e. Numerator is perfect derivative of denominator.

$\therefore \int \frac{3e^x}{3e^x + 12} \, dx = \log |3e^x + 12| + c$

Type No. 2 - Integrals of The Form : $\int \frac{ae^x + b}{ce^x + d}\, dx$

Then split the numerator such that,

Numerator = A (denominator) + B $\frac{d}{dx}$ (Denominator) and use type No. 1

e.g. : $\int \frac{4e^x - 25}{2e^x - 5}\, dx$

Put $4e^x - 25 = A\,(2e^x-5) + B\,\frac{d}{dx}(2e^x-5)$

$4e^x x-25 = 2\,Ae^x-5A + 2Be^x - 0$

$4e^x-25 = (2A + 2B)\,e^x - 5A$

$\Rightarrow -5\,A = -25 \qquad 2A + 2B = 4$

$\therefore\ A = 5 \qquad A + B = 2$

$B = 2 - A = 2-5$

$B = -3$

$$\therefore \int \frac{4e^x-25}{2e^x-5}\, dx = \int \frac{5(2e^x-5) -3(2e^x)}{2e^x-5}\, dx$$

$$= 5\int \frac{(2e^x-5)}{(2e^x-5)}\, dx - 3\int \frac{2e^x}{2e^x-5}$$

$$= 5\int 1dx - 3\ log\left|\,2e^x-5\,\right| + c$$

$$= 5x - 3\ \log\left|\,2e^x - 5\,\right| + c$$

Results :

1) $\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\, log\left|\frac{x - a}{x + a}\right| \,| + c$

2) $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a}\, log\left|\frac{a + x}{a - x}\right| + c$

3) $\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = log\left|\,x + \sqrt{x^2 + a^2}\,\right| + c$

4) $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = log\left|\,x + \sqrt{x^2 - a^2}\,\right| + c$

5) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\, log\left|\,x + \sqrt{x^2 - a^2}\,\right|$

6) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\, log\left|\,x + \sqrt{x^2 + a^2}\,\right| + c$

Examples :

1) $\int \frac{1}{16x^2 - 25}\, dx$

$= \int \frac{1}{(4x)^2 - (5)^2}\, dx$

$= \frac{1}{4}\left[\frac{1}{2(5)} \log \left| \frac{4x-5}{4x+5} \right|\right] + c$

$= \frac{1}{40} \log \left| \frac{4x-5}{4x+5} \right| + c$

2) $\int \frac{1}{36 - 9x^2}\, dx$

$= \int \frac{1}{(6)^2 - (3x)^2}\, dx$

$= \frac{1}{3}\left[\frac{1}{2(6)} \log \left| \frac{6 + 3x}{6 - 3x} \right|\right] + c$

$= \frac{1}{36} \log \left| \frac{6 + 3x}{6 - 3x} \right| + c$

3) $\int \frac{1}{\sqrt{49x^2 + 64}}\, dx$

$= \int \frac{1}{\sqrt{(7x)^2 + (8)^2}}\, dx$

$= \frac{1}{7} \log \left| 7x + \sqrt{(7x)^2 + (8)^2} \right| + c$

$= \frac{1}{7} \log \left| 7x + \sqrt{49^2 + 64} \right| + c$

4) $\int \frac{1}{\sqrt{4x^2 - 9}}\, dx$

$= \int \frac{1}{\sqrt{(2x)^2 - (3)^2}}\, dx$

$= \frac{1}{2} \log \left| 2x + \sqrt{(2x)^2 - (3)^2} \right| + c$

$= \frac{1}{2} \log \left| 2x + \sqrt{4x^2 - 9} \right| + c$

5) $\int \sqrt{x^2 - 9}\ dx = \int \sqrt{x^2 - (3)^2}\ dx$

$$= \frac{x}{2}\sqrt{x^2 - 9} + \frac{(3)^2}{2} \log \left| x + \sqrt{x^2 - 9} \right| + c$$

$$= \frac{x}{2}\sqrt{x^2 - 9} + \frac{9}{2} \log \left| x + \sqrt{x^2 - 9} \right| + c$$

Type No. 3 : Integrals of The Type :

$$\int \frac{1}{ax^2 + bx + c}\, dx \text{ or } \int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx \quad \text{or } \int \sqrt{ax^2 + bx + c}\, dx$$

Then use completing the square method

Step : 1 Make the coefficient of x^2 as one if it is not (Taking 'a' common From denominator)

i.e. $\frac{1}{a} \frac{1}{x^2 + \frac{b}{a}x + \frac{c}{a}}$

Step : 2 Add and Subtract the square of the half of coefficient of x

i.e. $\left(\frac{b}{2a}\right)^2$

$$\therefore \frac{1}{a} \int \frac{1}{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2}\, dx$$

$$= \frac{1}{a} \int \frac{1}{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2}\, dx$$

After this use the following result, $\int \frac{1}{x \pm a^2}\, dx$ or $\int \frac{1}{a^2 - x^2}\, dx$

Example :

1) $\int \frac{1}{4x^2 - 20 + 17}\, dx$

$$= \frac{1}{4} \int \frac{1}{x^2 - 5x + \frac{17}{4}}\, dx$$

Add and subtract $\left(\frac{1}{2} \times 5\right)^2 = \frac{25}{4}$

$$= \frac{1}{4} \int \frac{1}{x^2 - 5x + \frac{25}{4} + \frac{17}{4} - \frac{25}{4}}\, dx$$

$$= \frac{1}{4} \int \frac{1}{\left(x - \frac{5}{2}\right)^2 - 2}\, dx$$

$$= \frac{1}{4} \int \frac{1}{\left(x-\frac{5}{2}\right)^2-\left(\sqrt{2}\right)^2}\, dx$$

$$= \frac{1}{4} \times \frac{1}{2\sqrt{2}}\, log \left| \frac{\left(x-\frac{5}{2}\right)-\left(\sqrt{2}\right)}{\left(x-\frac{5}{2}\right)+\left(\sqrt{2}\right)} \right| + c$$

$$= \frac{1}{8\sqrt{2}}\, log \left| \frac{2x-5-2\sqrt{2}}{2x-5+2\sqrt{2}} \right| + c$$

Example :

2) $\int \frac{1}{\sqrt{4x^2-20x+17}}\, dx$

$$= \frac{1}{2} \int \frac{1}{\sqrt{x^2-5x+\frac{17}{4}}}\, dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{\left(x-\frac{5}{2}\right)^2-\left(\sqrt{2}\right)^2}}\, dx$$

$$= \frac{1}{2}\, log \left| \left(x-\frac{5}{2}\right) + \sqrt{\left(x-\frac{5}{2}\right)^2-\left(\sqrt{2}\right)^2} \right|$$

$$= \frac{1}{2}\, log \sqrt{x^2-5x+\frac{17}{4}} \Big| + c$$

Example :

3) $\int \sqrt{4x^2-20x+17}\, dx$

$$= 2 \int \sqrt{x^2-5x+\frac{17}{4}}\, dx$$

$$= 2 \int \sqrt{\left(x-\frac{5}{2}\right)^2-\left(\sqrt{2}\right)^2}\, dx$$

$$= 2 \left\{ \frac{\left(x-\frac{5}{2}\right)}{2} \sqrt{\left(x-\frac{5}{2}\right)^2-\left(\sqrt{2}\right)^2} + \frac{2}{2}\, log \left| \left(x-\frac{5}{2}\right) + \sqrt{\left(x-\frac{5}{2}\right)^2-\left(\sqrt{2}\right)^2} \right| \right\} + c$$

$$= \left(x-\frac{5}{2}\right) \sqrt{x^2-5x+\frac{17}{4}} \Big| + 2\ \ log \left| \left(x-\frac{5}{2}\right) + \sqrt{x^2-5x+\frac{17}{4}} + c \right.$$

Type No. 4 : Integration By Parts :

If integrand contains product of two function, then use integration is *by* parts.

$$\int uvdx = u\int vdx - \int \left(\left[\int vdx\right] \frac{d}{dx}(u) \right) dx$$

L $\downarrow$ Log, A $\downarrow$ Algebraic, E $\downarrow$ Exponential

Selection of v :

1) First preference is given to exponential function e.g. e^x, a^x
2) Second preference is given to algebraic function e.g. x, x^3 etc.
3) Third preference is given to constant function i.e. 1 (one).

Example :

$$\int xe^x dx$$

$$= x\int e^x dx - \int \left[\int e^x\, dx\right] \frac{d}{dx}(x)dx$$

$$= xe^x - \int e^x(1)\, dx + c,$$

$$= xe^x - \int e^x\, dx + c,$$

$$= xe^x - e^x + c$$

Type No : 5 : Integrals of The Type :

$$\int e^x\, [f(x) + f'(x)]\, dx = e^x f(x) + c$$

Example : $\int e^x \left(\frac{x\log x+1}{x}\right) dx$

$$= \int e^x \left[\frac{x\log x}{x} + \frac{1}{x}\right] dx$$

$$= \int e^x \left[\log x + \frac{1}{x}\right] dx$$

$$= e^x \log x + c$$

Type No. 6 : Integration by Partial Fractions :

	Rational Form	**Partial Form**
1)	$\frac{px \pm q}{(x-a)\,(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2)	$\frac{px^2 \pm qx \pm r}{(x-a)\,(x-b)\,(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$

	Rational Form	Partial Form
3)	$\dfrac{px \pm q}{(x-a)^2}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-a)^2}$
4)	$\dfrac{px^2 \pm qx \pm r}{(x-a)^2\ (x-b)}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-b)}$
5)	$\dfrac{px^2 \pm qx \pm r}{(x-a)\ (ax^2 \pm bx+c)}$	$\dfrac{A}{(x \pm a)}+\dfrac{Bx+C}{ax^2 \pm bx \pm c}$

Example :

1) $I = \int \dfrac{x^2+2}{(x-1)(x+2)(x+3)}\, dx$

Consider $\dfrac{x^2+2}{(x-1)(x+2)(x+3)} = \dfrac{A}{(x-1)}+\dfrac{B}{(x+2)}+\dfrac{C}{(x+3)}$

Put $x = 1$,we get $A = \dfrac{1}{4}$

$x = -2$, we get $B = -2$ and $x = -3$, we get $C = \dfrac{11}{4}$

$$I = \int \frac{\frac{1}{4}}{(x-1)} + \frac{(-2)}{(x+2)}\ \frac{\frac{11}{4}}{(x+3)}\ dx$$

$$I = \frac{1}{4}\int \frac{1}{(x-1)}\ dx - 2\int \frac{1}{(x+2)} dx + \frac{11}{4}\int \frac{1}{(x-1)}\ dx$$

$$= \frac{1}{4}\ log\ |\ x-1|\ -2\ log\ |\ x+2|\ + \frac{11}{4}\ log\ |x+3|\ + c$$

Activity 1: $\int \dfrac{x-1}{(x-3)(x-2)}\ dx$

Now $\dfrac{x-1}{(x-3)(x-2)} = \dfrac{[\quad]}{(x-3)} + \dfrac{[\quad]}{(x-2)}$

$\therefore\ x-1= (x-2)\ A + (x-3)B$

put $x = 3$ $\qquad \therefore\ A = \boxed{\quad}$

put $x = 2$ $\qquad \therefore\ B = \boxed{\quad}$

$$\therefore \int \frac{(x-1)}{(x-3)(x-2)}\ dx = \int \frac{2}{(x-3)}\ dx + \int \frac{-1}{(x-2)}\ dx$$

$$= 2log\ |x-3|-log\ |x-2|+ c$$

Activity 2 : Let = $\int e^x \sqrt{e^{2x}+1}\, dx$

put $e^x = t \therefore e^x dx = dt$

$\therefore$ I = $\int \sqrt{t^2+1}\, dt$

$= \frac{\boxed{}}{2} \sqrt{t^2+1} + log \left| t + \sqrt{t^2+1} \right| + c$

$= \frac{e^x}{2} \sqrt{\boxed{}+1} + [\ ]\ log \left| e^x + \sqrt{e^{2x}+1} \right| + c$

Ans : 1) A, B, 2, -1 2) e^x, t, $\frac{1}{2}$, e^{2x}

PROBLEMS FOR PRACTICE

1) MCQ :

i) $\int \frac{(x+2)}{2x^2+6x+5} dx = P \int \frac{(4x+6)}{2x^2+6x+5} dx + \frac{1}{2} \int \frac{1}{2x^2+6x+5} dx$ *then* $P =$

a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) $\frac{1}{4}$ d) 2

ii) $\int \frac{1}{(x-x^2)} dx =$

a) $log\ x - log|1-x| + c$ b) $log|1-x^2| + c$

c) $-log\,x + log|1-x| + c$ d) $log(x - x^2) + c$

iii) $\int \frac{e^{2x}+e^{-2x}}{e^x} dx =$

a) $e^x - \frac{1}{3e^{3x}} + c$ b) $e^x x + \frac{1}{3e^{3x}} + c$

c) $e^{-x} + \frac{1}{3e^{3x}} + c$ d) $e^{-x} + \frac{1}{e^x} + c$

2) Fill in the blanks :

1) $\int \frac{x^2+x-6}{(x-2)(x-1)} dx = x +$...................... $+ c$

2) If $f'(x) = \frac{1}{x} + x$ *and* $f(1) = \frac{5}{2}$ Then $f(x) = log\,x + \frac{x^2}{2} +$ ....

3) $\int \frac{1}{x^3} [log\, x^x]^2\ dx = P\,(log x)^3 + c$ *then* $P =$ ....

3) State whether each of the following is true or False.

1) If $\int x\, f(x)dx = \dfrac{f(x)}{2}$ then $f(x) = e^{x^2}$

2) For $\int \dfrac{x-1}{(x+1)^3}\ e^x\ dx = e^x\ f(x) + c$ then $f(x) = (x+1)^2$

3) If $\int \dfrac{(x+1)}{(x+1)(x-2)}\ dx = A\ log\ |\ x+1| + B\ log\ |\ x-2|\ then\ A + B = 1$

Evaluate :

1) $\int \left[\dfrac{1}{(6x + 5)^4} - \dfrac{1}{(8 - 3x)^9}\right] dx$

2) $\int \dfrac{1}{\sqrt{x} + \sqrt{x-7}}\ dx$

3) $\int x^3\ (2 - \dfrac{3}{x})^2\ dx$

4) $\int \dfrac{dx}{(xlogx.log(logx)}$

5) $\int \dfrac{dx}{1+e^{-x}}$

6) $\int \dfrac{3x^2}{\sqrt{1+x^3}}\ dx$

7) $\int \dfrac{3e^x+4}{2e^x-B}\ dx$

8) $\int \dfrac{1}{9x^2-4}\ dx$

9) $\int \dfrac{1}{\sqrt{4x^2-9}}\ dx$

10) $\int \log x\ dx$

Four marks examples : Evaluate :

1) $\int \dfrac{1}{x\sqrt{(logx)^2-5}}\ dx$

2) $\int \dfrac{1}{2x^2 + x - 1}\ dx$

3) $\displaystyle\int \frac{e^x}{(e^{2x}+6e^x+5)}\,dx$

4) $\displaystyle\int \frac{1}{\sqrt{x^2-8x-20}}\,dx$

5) $\int x^3.\log x\ dx$

6) $\int x^2.e^{3x}dx$

7) $\sqrt{x^2-4x-5}\ dx$

8) $\displaystyle\int e^x\,\frac{(1+x^2)}{(1+x)^2}\,dx$

9) $\displaystyle\int \frac{\log x}{x(1+\log x).(2+\log x)}\,dx$

10) $\displaystyle\int \frac{3x-2}{(x+1)^2.(x+3)}\,dx$

■■■

6. Definite Integration

Fundamental theorem of integral calculus :

Let f be the continuous function defined on $[a, b]$ and if $\int f(x)\,dx = g(x) + c$ then

$$\int_a^b f(x)\,dx = [g(x) + c]_a^b$$

$$= [(g(b) + c) - (g(a) + c)]$$

$$= g(b) + c - g(a) - c$$

$$= g(b) - g(a)$$

$$\boxed{\text{Thus } \int_a^b f(x)\,dx = g(b) - g(a)}$$

In $\int_a^b f(x)\,dx$, a is called lower limit and b is called as an upper limit.

There is no need of taking the constant of integration c, because it gets eliminated.

Solved Examples :

Ex. 1) Evaluate: $\int_1^2 x^2\,dx$

Solution : $\int_1^2 x^2\,dx. = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

Ex. 2) Evaluate $\int_2^3 \frac{x^2}{(x+2)(x+3)}\,dx$

Solution : $\frac{x^2}{(x+2)(x+3)} = \frac{A}{(x+2)} = \frac{B}{(x+3)}$

$\therefore\ x = A(x+3) + B(x+2)$........ (i)

Put $x = -2$, we get $A = -2$ and Put $x = -3$, we get $A = 3$

$\therefore\ \frac{x^2}{(x+2)(x+3)} = \frac{-2}{(x+2)} = \frac{3}{(x+3)}$

$\therefore\ \int_2^3 \frac{x^2}{(x+2)(x+3)}\,dx = -2\int_2^3 \frac{1}{(x+2)}\,dx + 3\int_2^3 \frac{1}{(x+3)}\,dx$

$= -2\ |log\,(x+2)|_2^3 + 3\ |log\,(x+3)|_2^3$

$= -2\,(log\,5 - log\,4) + 3\,(log\,6 - log5)$

$= -5\ log\,5 + 2\ log\,4 + 3\ \log 6$

$= \log\left(\frac{4^2.6^3}{5^5}\right) = log\left(\frac{3456}{3125}\right)$

Properties of definite integrals :

Property 1 : $\int_a^a f(x)\,dx = 0$

Property 2 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Property 3 : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Property 4 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$

Property 5 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Property 6 : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Property 7 : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Property 8 : $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(x)$ is even function.

$= 0$, if $f(x)$ is odd function.

Note : $f(x)$ even function if $f(-x) = f(x)$ and $f(x)$ odd function if $f(-x) = -f(x)$

Ex. 3) Evaluate $\int_{-7}^{7} \frac{x^3}{x^2+7}\,dx$

Solution : Let $I = \int_{-7}^{7} \frac{x^3}{x^2+7}\,dx$

Let $f(x) = \frac{x^3}{x^2+7}$

$\therefore f(-x) = \frac{(-x)^3}{(-x)^2+7} = -\frac{x^3}{x^2+7} = -f(x)$

$\therefore f(x)$ is an odd function $\therefore \int_{-7}^{7} \frac{x^3}{x^2+7}\,dx = 0$

Ex. 4) Evaluate $\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5} \quad \sqrt[3]{9-x}} + dx$

Solution : Let $I = \int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5} \quad \sqrt[3]{9-x}}\,dx$ .... (i)

By property $\int_0^a f(x)\,dx = \int_0^a f(a+b-x)\,dx$

$$\therefore\ I = \int_1^3 \frac{\sqrt[3]{(4-x)+5}}{\sqrt[3]{(4-x)+5} + \sqrt[3]{9-(4-x)}}\, dx$$

$$\therefore\ I = \int_1^3 \frac{\sqrt[3]{9-x}}{\sqrt[3]{9-x} + \sqrt[3]{x+5}}\, dx \ \text{..................} \ \text{(ii)}$$

Adding (i) and (ii)

$$\therefore\ 2I = \int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5} \quad \sqrt[3]{9-x}}\, dx + \int_1^3 \frac{\sqrt[3]{9-x}}{\sqrt[3]{9-x} + \sqrt[3]{x+5}}\, dx$$

$$\therefore\ 2I = \int_1^3 \frac{\sqrt[3]{x+5} + \sqrt[3]{9-x}}{\sqrt[3]{x+5} + \sqrt[3]{9-x}}\, dx$$

$$\therefore\ 2I = \int_1^3 1\ dx$$

$$\therefore\ 2I = |x|_1^3 = (3-1) = 2$$

$I = 1$

$$\therefore\ \int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5} + \sqrt[3]{9-x}}\, dx = 1$$

Questions for Practice

1) Choose the correct alternative.

1) $\int_1^3 \frac{x^3}{4-x^2}\, dx =$ ....

a) 9 b) –9 c) $\frac{9}{2}$ d) 0

2) $\int_4^9 \frac{dx}{\sqrt{x}}\, dx =$ ....

a) 1 b) 2 c) $\frac{5}{2}$ d) 3

3) If $\int_0^a 3x^2\, dx = 8$ then $a =$ ....

a) 1 b) 2 c) 3 d) 0

2) Fill in the blanks.

1) If $\int_0^a (2x+1)\, dx = 2$, then $a =$ ....

2) $\int_2^4 \frac{x}{x^2+1}\,dx =$ ....

3) $\int_0^2 e^x dx =$ ....

3) State whether each of the following is True or False.

1) $\int_0^1 \frac{1}{2x+5}\,dx = \frac{1}{2}\log\left(\frac{7}{5}\right)$

2) $\int_0^1 e^{2x} dx = \frac{1}{2}(e^2 - e)$

3) $\int_{-5}^5 \frac{x^5}{4-x^2}\,dx = 5$

4) Solve The following.

1) Evaluate $\int_1^2 \frac{3x}{(9x^2-1)}\,dx$

2) Evaluate $\int_0^1 \frac{x^2+3x+2}{\sqrt{x}}\,dx$

3) Evaluate $\int_0^1 \frac{dx}{\sqrt{1+x}+\sqrt{x}}$

4) Evaluate $\int_2^7 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx$

5) Evaluate $\int_4^7 \frac{(11-x)^2}{x^2+(11-x)^2}\,dx$

6) Evaluate $\int_0^1 x(1-x)^5\,dx$

7) Evaluate $\int_1^3 \log x\,dx$

8) Evaluate $\int_1^2 \frac{1}{(x+1)(x+3)}\,dx$

9) Evaluate $\int_1^2 \frac{x+3}{x(x+2)}\,dx$

10) Evaluate $\int_0^4 \frac{1}{\sqrt{x^2+2x+3}}\,dx$

Activities

1) Evaluate $\int_1^2 \frac{1}{x^2+6x+5}\,dx$

Solution : Let $I = \int_1^2 \frac{1}{x^2+6x+5}\,dx$

$= \int_1^2 \frac{1}{(x+3)^2 - \boxed{}}\,dx$

$$= \frac{1}{2}\left|\log\left(\frac{x+1}{\boxed{}}\right)\right|_1^2$$

$$= \frac{1}{4}\left[log\left(\boxed{}\right) - log\left(\frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\, log\left(\frac{9}{\boxed{}}\right)$$

2) **Evaluate** $\int_1^3 x^2 . \log x \, dx$

Solution : Let $I = \int_1^3 x^2 . \log x \; dx$

$$= \int_1^3 \int (log\ x).(x^2)\ dx$$

$$= \left| (log\ x).\int x^2 dx \right|_1^3 - \int_1^3 \left\{ \frac{d}{dx} log x . \int x^2\ dx \right\} dx$$

$$= \left| (log\ x).\boxed{} \right|_1^3 - \frac{1}{3}\int_1^3 x^2 dx$$

$$= \frac{1}{3}\left|\boxed{}\right|_1^3 - \frac{1}{9}\left| x^3 \right|_1^3$$

$$= \frac{1}{3}\left[\boxed{} - log 1\right] - \frac{26}{9}$$

$$= \boxed{}$$

■■■

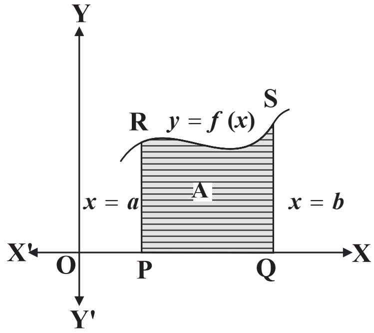
7. Application of Definite Integration

Geometrically $\int_a^b f(x)\ dx$ gives the area A under the curve $y = f(x)$ with $f(x) > 0$ and bounded by the X-axis and the lines $x = a$, $x = b$; and is given by

$\int_a^b f(x)\ dx = \Phi(b) - \Phi(a)$, where $\int f(x)\ dx = \Phi(x)$

This is also known as fundamental theorem of integral calculus.

We shall find the area under the curve by using definite integral.



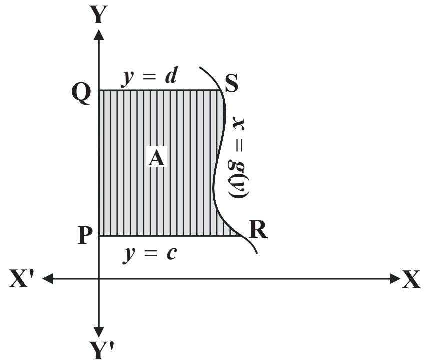
Area under a curve :

The curve $y = f(x)$ is continuous in $[a, b]$ and $f(x) \geq 0$ in $[a, b]$.

The area shaded in figure is bounded by the curve $y = f(x)$, X-axis and the lines $x = a$, $x = b$ and is given by the definite integral $\int_{x=a}^{x=b} (y)\ dx$

A = area of the shaded region.

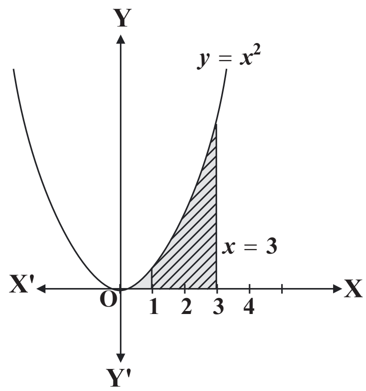
$A = \int_a^b f(x)\ dx$



The area A, bounded by the curve $x = g(y)$, Y axis and the lines $y = c$ and $y = d$ is given by

$A = \int_{y=c}^{y=d} x\ .dy \ = \int_{y=c}^{y=d} g(y)\ dx$

Ex. 1) Find the area bounded by the curve $y = x^2$, the Y-axis, the X-axis and $x = 3$



Solution : The required area

$A = \int_{x=0}^{x=3} y\ dx. = \int_a^b x^2 dx. = \left[\frac{x^3}{3}\right]_0^3$

$A = 9 - 0 \qquad = 9$ *sq.unit.*

Ex. 2) Find the area of the regions bounded by the following curve, the X-axis and the given lines :

i) $y = x^2$, $x = 1$, $x = 2$ ii) $y^2 = 4x$, $x = 1$, $x = 4$, $y \geq 0$

Solution : 1) Let A be the required area

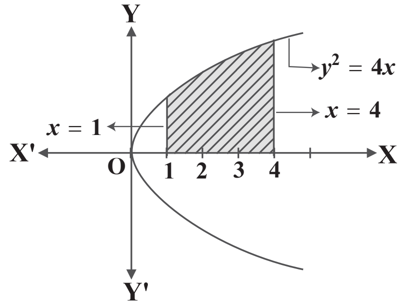
$A = \int_1^2 y\ dx$

$= \int_1^2 x^2 \, dx$

$= \frac{1}{3} \left[x^3\right]_1^2$

$= \frac{1}{3} [8 - 1]$

$A = \frac{7}{3}$ sq.unit.



ii) $A = \int_1^4 y \, dx = \int_1^4 2\sqrt{x} \, dx$

$= 2\left[\frac{2}{3} x^{\frac{3}{2}}\right]_1^4 = \frac{1}{4}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{28}{3}$

$\therefore A = \frac{28}{3}$ sq. unit.

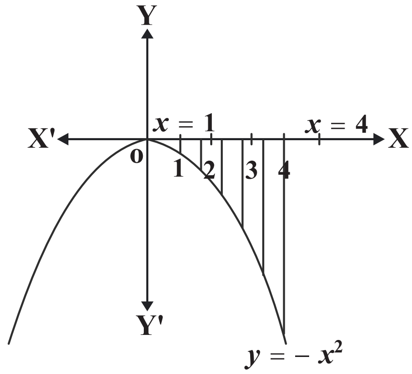
Ex. 3) Find the area bounded by the curve $y = -x^2$, X-axis and lines $x = 1$ and $x = 4$.

Solution : Let A be the area bounded by the curve $y = -x^2$, X-axis and $1 \le x \le 4$.

$I = \int_1^4 y \, dx = \int_1^4 [-x^2] \, dx$

$= \left[-\frac{x^3}{3}\right]_1^4 = -\frac{64}{3} + \frac{1}{3} = -21$

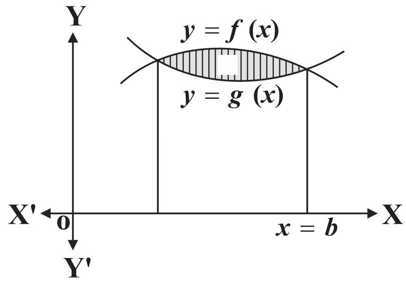
The required area = A = 21 sq. unit.



Thus, if $f(x) \le 0$ or $f(x) \ge 0$ in $[a, b]$ then the area enclosed between $y = f(x)$, X-axis and $x = a$, $x = b$ is $\left|\int_a^b f(x) \, dx\right|$

Area between two curves :

Let $y = f(x)$ and $y = g(x)$ be the equations of the two curves as shown in fig.



Let A be the area bounded by the curves $y = f(x)$ and $y = g(x)$

$A = |A_1 - A_2|$

Where A_1 = Area bounded by the curve $y = f(x)$, X-axis and $x = a$, $x = b$.

A_2 = Area bounded by the curve $y = g(x)$, X-axis and $x = a$, $x = b$.

The point of intersection of the curves $y = f(x)$ and $y = g(x)$ can be obtained by solving their equations simultaneously.

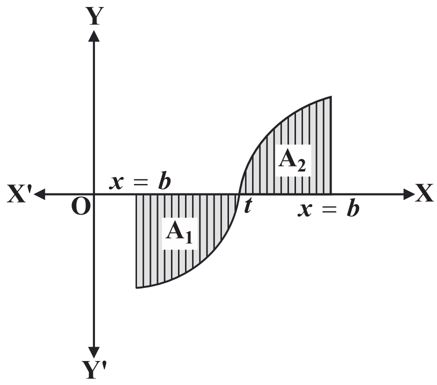
$\therefore$ The required area $A = \left|\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right|$

If the area A is divided into two parts A_1 and A_2 such that A_1 is the part of $a \leq x \leq t$ where $f(x) \leq 0$ and

A_2 is the part of $a \leq x \leq t$ where $f(x) \geq 0$ then in A_1, the required area is below the X-axis and in A_2, the required area is above the X-axis.

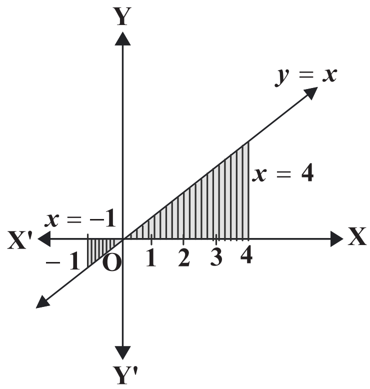
Now the total area $A = A_1 + A_2$

$= \left|\int_a^t f(x)\,dx\right| + \left|\int_t^b f(x)\,dx\right|$



Ex. 4) Find the area bounded by the line $y = x$, X-axis and the lines $x = -1$ and $x = 4$.

Solution : Consider the area A, bounded by straight line $y = x$, X-axis and $x = -1$, $x = 4$. From given figure, A is divided into A_1 and A_2



The required area $A_1 = \int_{-1}^{0} y\,dx = \int_{-1}^{0} x\,dx$

$$= \left[\frac{x^2}{2}\right]_{-1}^{0} = 0 - \frac{1}{2} = -\frac{1}{2}$$

But area is always positive.

$\therefore A_1 = \left|-\frac{1}{2}\right|$ sq. unit $= \frac{1}{2}$ sq unit.

$$A_2 = \int_0^4 y\,dx = \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = \frac{4^2}{2} = 8 \text{ square unit.}$$

Required area $A = A_1 + A_2 = \frac{1}{2} + 8 = \frac{17}{2}$ square unit

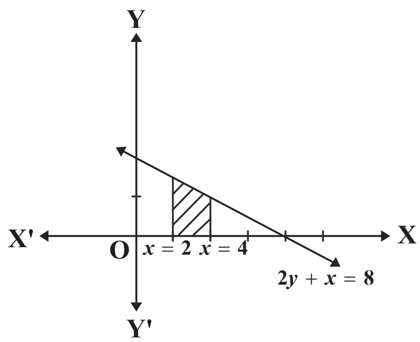
Ex. 5) Using integration, find the area of the region bounded by the line $2y + x = 8$, X-axis and the lines $x = 2$ and $x = 4$.

Solution : The required region is bounded by the lines $2y + x = 8$ and $x = 2$, $x = 4$ and X-axis.

$\therefore\ y = \frac{1}{2}(8 - x)$ and the limits are $x = 2$, $x = 4$.

Required area = Area of the shaded region

$$= \int_2^4 y\,dx = \int_2^4 \frac{1}{2}(8 - x)\,dx$$

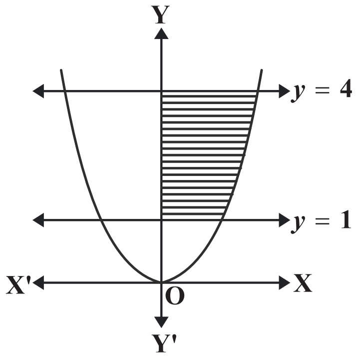


$$= \frac{1}{2}\left[8x - \frac{x^2}{2}\right]_0^4$$

$$= \frac{1}{2}\left[\left(8.(4) - \frac{4^2}{2}\right) - \left(8.(2) - \frac{2^2}{2}\right)\right]$$

$= 5$ sq.unit

Ex. 6) Find the area of the region bounded by the curves $x^2 = 16y$, $y = 1$, $y = 4$ and the Y-axis, lying in the first quadrant.



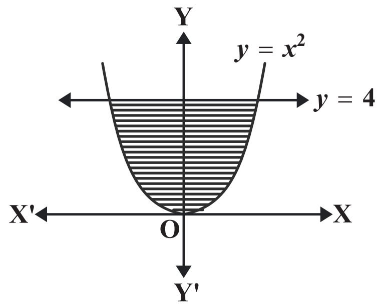
Solution : Required area $= \int_1^4 x\, dx = \int_1^4 \sqrt{16y}\; dy$

$$= \int_1^4 4\sqrt{y}\; dy$$

$$= 4\left[\frac{2}{3}y^{\frac{3}{2}}\right]_1^4 = \frac{8}{3}[8 - 1]\ \frac{56}{3}$$

$$A = \frac{56}{3} \text{ sq. unit.}$$

Ex. 7) Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.



Solution : Required area A = 2 × area of OPQO

$$A = 2\int_0^4 x\; dy = 2\int_0^4 \sqrt{y}\; dy = 2 \times \frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^4$$

$$= \frac{4}{3}\left[4^{\frac{3}{2}} - 0\right] = \frac{32}{3}$$

$\therefore$ Required area $= \frac{32}{3}$ sq. unit

Questions for Practice

1) Choose the correct option from the given alternative.

1) The area bounded by the curve $y = x^2$, the X-axis and the lines $x = 1$ and $x = 3$ is

a) $\frac{26}{3}$ sq. unit b) $\frac{3}{26}$ sq. unit c) 26 sq. unit d) 3 sq. unit

2) The area under the curve $= 2\sqrt{x}$, enclosed between the lines $x = 0$ and $x = 1$ is

a) 4 sq. unit b) $\frac{3}{4}$ sq. unit c) $\frac{2}{3}$ sq. unit d) $\frac{4}{3}$ sq. unit

2) Fill in the blanks.

1) The area of the region bounded by the curve $y = 2x$, X-axis and lines $x = 0$, $x = 5$ is ....

2) The area bounded by the curve $x = 2y$ and lines $y = 0$, $y = 4$ is ....

3) True or False.

1) The area of the portion lying above the X-axis is positive.

2) The area bounded by the curve $y = 2x$, the X-axis and the lines $x = 1$ and $x = 3$ is 2.

4) Solve The following.

1) Find the area of the region bounded by the following curves, X-axis and the given lines.

i) $y = x^2 + 1$, $x = 2$ and $x = 4$

ii) $y = 2x + 3$, $x = 0$ and $x = 3$

iii) $y^2 = x$, $x = 0$, $x = 4$

iv) $y^2 = 16x$ and $x = 0$, $x = 4$

2) Find the area of the region bounded by the curve $y^2 = 25x$, and the line $x = 5$.

3) Find the area of the region bounded by the curve $y = x^2$ and the line $y = 10$.

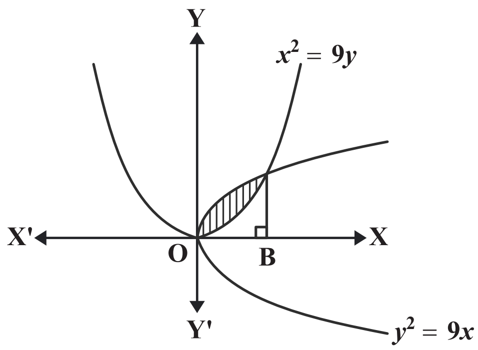
5) Find the area of the region bounded by the curves $y^2 = 9x$ and $x^2 = 9y$

Solution : The equations of the curves are

$y^2 = 9x$ ....(i)

$x^2 = 9y$ .... (ii)

Solving equation (i) and (ii) , we get x and y.



$\therefore$ The points of intersection of the curves are (0, 0) and $\left(\boxed{}\right)$

$\therefore$ Required area

$$A = \int_0^9 3\sqrt{x}\,dx - \int_0^9 \boxed{}\,dx$$

$$A = 3\left|\boxed{}\right|_0^9 - \frac{1}{9}\left|\frac{x^3}{3}\right|_0^9$$

$$A = \boxed{} \text{ sq units}$$

■■■

8. Differential Equation

Points to be remembered :

1) An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.

2) The order of highest derivative occurring in the differential equation is called order of the differential equation. The power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any is called degree of the differential equation

3) A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.

4) To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.

> **Note :**
>
> 1) To find the degree of the differential equation, we need to have a positive integer as the index of each derivative.
>
> 2) The Order and degree (if defined) of a differential equation are always positive integers.
>
> 3) The order of a differential equation representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.
>
> 4) We shall prefer to use the following notations for derivatives :
> $\frac{dy}{dx} = y'$, $\frac{d^2y}{dx^2} = y''$, $\frac{d^3y}{dx^3} = y'''$ and so on.
>
> 5) For derivatives of higher order, it will be inconvenient to use so many dashes as super suffix therefore, we use the notation y_n for nth order derivative $\frac{d^m y}{dx^m}$.

5) Variable separable method is used to solve such an equation in which variables can be separated completely i.e. terms containing y should remain with dy and terms containing x should remain with dx.

6) If the homogeneous differential equation is in the form $\frac{dx}{dy} = F(x,y)$ where, $F(x,y)$ is homogenous function of degree zero, then we make substitution $\frac{y}{x} = v$ i.e. $y = vx$ and we proceed further to find the general solution by writing $\frac{dx}{dy} = F(x,y) = g\left(\frac{y}{x}\right)$.

7) If the homogeneous differential equation is in the form $\frac{dx}{dy} = F(x,y)$ where, $F(x,y)$ is homogenous function of degree zero, then we make substitution $\frac{y}{x} = v$ i.e. $y = vy$ and we proceed further to find the general solution by writing $\frac{dx}{dy} = F(x,y) = g\left(\frac{x}{y}\right)$.

8) The most general form of a linear differential equation of the first order is : $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only, is known as a first order linear differential equation.

Steps involved to solve first order linear differential equation :

(i) Write the given differential equation in the form : $\frac{dy}{dx} + Py = Q$
where P, Q are constants or functions of x only.

(ii) Find the Integrating Factor (I.F) = $e^{\int p\,dx}$.

(iii) Write the solution of the given differential equation as $y\,(I.F) = \int [Q \times (I.F.)]\, dx + C$

9) Application of differential Equations
Differential equations can be used to describe mathematical models such as population expansion or radioactive decay etc. Some applications of differential equation are :

a) Population Growth b) Growth of Bacteria c) Radio Active Decay

Solved Examples :

Ex. 1) Find the order and degree, if defined, of the following differential equations :

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = \sqrt{1 + \frac{d^3y}{dx^3}}$$

Solution : This equation expressed as $\left(\frac{d^2y}{dx^2} + \frac{dy}{dx} + x\right)^2 = 1 + \frac{d^3y}{dx^3}$

The highest order derivative present in the given differential equation is $\frac{d^3y}{dx^3}$, so its order is three. It is a polynomial equation in $\frac{d^3y}{dx^3}$, $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and the highest power raised to $\frac{d^3y}{dx^3}$ is one, so its degree is one.

Ex. 2) Obtain the differential equation form the following : $y = c_1e^{3x} + c_2e^{2x}$

Solution : $y = c_1e^{3x} + c_2e^{2x}$ .... (I)

Differentiate *w.r.t.x*, we get

$\frac{dy}{dx} = 3c_1e^{3x} + 2c_2e^{2x}$ .... (II)

Again differentiate *w.r.t.x*, we get

$\frac{d^2y}{dx^2} = 9c_1e^{3x} + 4c_2e^{2x}$ .... (III)

As equations (I), (II) and (III) in c_1e^{3x} and c_2e^{2x} are consistent

$$\begin{vmatrix} y & 1 & 1 \\ \frac{dy}{dx} & 3 & 2 \\ \frac{d^2y}{dx^2} & 9 & 4 \end{vmatrix} = 0$$

$$\therefore\ y\ (12 - 18) -1\left(4\frac{dy}{dx} -2\frac{d^2y}{dx^2}\right) + 1\left(9\frac{dy}{dx} -3\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore\ -6y - 4\frac{dy}{dx} + 2\frac{d^2y}{dx^2} + 9\frac{dy}{dx} - 3\frac{d^2y}{dx^2} = 0$$

$\therefore\ -\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$ is the required differential equation.

Ex. 3) Verify that : $y = \log x + c$ is a solution of the differential equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

Solution : Here $y = \log x + c$

Differentiate *w.r.t.x*, we get, $\frac{dy}{dx} = \frac{1}{x}$

$\therefore\ x\frac{dy}{dx} = 1$, Differentiate *w.r.t.x*, we get

$$\therefore \; x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

Hence $y = \log x + c$ is a solution of the differential equation $x \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

Ex. 4) Find the general solution of the differential equation : $\boldsymbol{\dfrac{dt}{dx} = \dfrac{x\log x}{t}}$

Solution : $\dfrac{dt}{dx} = \dfrac{x\log x}{t}$

$$\therefore \; \frac{dx}{x\log x} = \frac{dt}{t}$$

Integrating both sides, we get

$$\therefore \int \frac{dx}{x\log x} = \int \frac{dt}{t}$$

$\therefore$ log (log x) = log (t) + log c

$\therefore$ log (log x) = log (tc)

$\therefore$ log x = ct $\quad \therefore$ $e^{ct} = x$

Ex. 5) Find the particular solution of the differential equation $\boldsymbol{\dfrac{y-1}{y+1} + \dfrac{x-1}{x+1} \cdot \dfrac{dy}{dx} = 0}$ **when $x = y = 2$**

Solution : $\dfrac{y-1}{y+1} + \dfrac{x-1}{x+1} \cdot \dfrac{dy}{dx} = 0$

$$\therefore \; \frac{x-1}{x+1}\,dx + \frac{y+1}{y-1}\,dy = 0$$

$$\therefore \; \frac{(x-1)+2}{x-1}\,dx + \frac{(y-1)+2}{y-1}\,dy = 0$$

$$\therefore \; \left(1 + \frac{2}{x-1}\right) dx + \left(1 + \frac{2}{y-1}\right) dy = 0$$

Integrating, we get

$$\therefore \int 1\,dx + 2\int \frac{1}{x-1}\,dx + \int 1\,dy + 2\int \frac{1}{y-1}\,dy = 0$$

$\therefore$ $x + 2\log(x-1) + y + 2\log(y-1) = c$

$\therefore$ $x + y + 2\log[(x-1)(y-1)] = c$ .... (I)

When $x = 2$, $y = 2$. So eq. (I) becomes

$\therefore\ 2+2+2\log\left[(2-1)\ (2-1)\right] = c$

$\therefore\ 4+2\log(1\ x\ 1) = c$

$\therefore\ 4+2\log(1) = c$

$\therefore\ 4+2(0) = c \quad \therefore\ c = 4$

Put in eq. (I), we get,

$\therefore\ x+y+2\log \therefore \left[(x-1)\ (y-1)\right] = 4$ is the required particular solution.

Ex. 6) Reduce the following differential equations to the separated variable form and hence find the Particular solution.

$$\frac{dy}{dx} = \frac{x+y+1}{x+y-1} \text{ when } x = \frac{2}{3} \text{ and } y = \frac{1}{3}$$

Solution : $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+y-1}$ .... (i)

Put $x+y=t$ .... (ii)

$\therefore\ y = t - x$

Differentiate *w.r.t.x*, we get

$\therefore\ \dfrac{dy}{dx} = \dfrac{dt}{dx} - 1$ .... (iii)

Put (ii) and (iii) in equation (i) we get,

$$\frac{dt}{dx} - 1 = \frac{t+1}{t-1}$$

$$\therefore\ \frac{dt}{dx} = \frac{t+1}{t-1} + 1$$

$$\therefore\ \frac{dt}{dx} = \frac{t+1+t-1}{t-1}$$

$$\therefore\ \frac{dt}{dx} = \frac{2t}{t-1}$$

$$\therefore\ \frac{t-1}{t}\, dt = 2\,dx$$

$$\therefore\ \left(1-\frac{1}{t}\right) dt = \int 2\,dx$$

Integrating both sides, we get

$$\int \left(1-\frac{1}{t}\right) dt = \int 2\,dx$$

$\therefore\ t - \log t = 2x + c$

$\therefore\ x + y - \log(x+y) = 2x + c$

$\therefore\ y - x - \log(x+y) = c$

$\therefore\ y-x-c = \log(x+y)$ .... (iv)

Putting $x = \dfrac{2}{3}$ and $y = \dfrac{1}{3}$, we get, $\therefore$ from .... (iv)

$\dfrac{1}{3} - \dfrac{2}{3} - c = \log\left(\dfrac{2}{3} + \dfrac{1}{3}\right)$

$\therefore\ \dfrac{-1}{3} - c = \log(1)$

$\therefore\ \dfrac{-1}{3} - c = 0 \quad \therefore\ c = \dfrac{-1}{3}$

$\therefore\ y-x-\left(\dfrac{-1}{3}\right)13 = \log(x+y)$

$\therefore\ y-x+\dfrac{1}{3}=\log(x+y)$ is the particular solution

Ex. 7) Solve the differential equations $\dfrac{dy}{dx} = \dfrac{y+\sqrt{x^2+y^2}}{x}$

Solution : $\dfrac{dy}{dx} = \dfrac{y+\sqrt{x^2+y^2}}{x}$ .... (I)

It is homogeneous differential equation

Put $y = vx$ .... (II)

Differentiate $w.r.t.x$, we get

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ .... (III)

Put (II) and (III) in equation (I) we get,

$v + x\dfrac{dv}{dx} = \dfrac{vx+\sqrt{x^2+y^2x^2}}{x}$

$\therefore\ v + x\dfrac{dv}{dx} = v + \sqrt{1+v^2}$

$\therefore\ x\dfrac{dv}{dx} = \sqrt{1+v^2}$

$\therefore\ \dfrac{dv}{\sqrt{1+v^2}} = \dfrac{dx}{x}$

Integrating both sides we get

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\therefore \log\left(v + \sqrt{1+v^2}\right) = \log x + \log c$$

$$\therefore \log\left(v + \sqrt{1+v^2}\right) = \log cx$$

$$\therefore v + \sqrt{1+v^2} = cx$$

$$\therefore \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = cx$$

$\therefore y + \sqrt{x^2+y^2} = cx^2$ is the solution.

Ex. 8) Solve the differential equations : $\frac{dy}{dx} = y + 2x$, where $y(0) = 0$

Solution : $\frac{dy}{dx} = y + 2x$

$$\therefore \frac{dy}{dx} - y = 2x$$

This is linear differential equation of the type $\frac{dy}{dx} + Py = Q$

It's solution is, $y(\text{I.F}) = \int [Q \times (\text{I.F.})]\, dx + C$ .... (I)

where, $(\text{I.F.}) = e^{\int P.dy} = e^{\int -1.dx}$

$(I.F.) = e^{-x}$

equation (I) becomes,

$$y \,.\, e^{-x} = \int 2xe^{-x} . dx + c$$

$y \,.\, e^{-x} = \int 2xe^{-x} . dx + c$ .... (II)

Consider, $\int x.\ e^{-x}\ .\ dx$

$$= x \int e^{-x}\,.\,dx - \int \left[1 \times \frac{e^{-x}}{-1}\right] . dx$$

$$= \frac{x . e^{-x}}{-1} + \int e^{-x}\,.\,dx$$

$$= x.\ e^{-x} + \int e^{-x}\,.\,dx$$

$$= x.\ e^{-x} - e^{-x}$$

equation (II) becomes,

$$y.\ e^{-x} = 2(-x.e^{-x} - e^{-x}) + c$$

$\therefore\ y.\ e^{-x} = -2xe^{-x} - 2e\ (-x) + c$ .... (III)

Given $y(0) = 0$

i.e. put $x = 0$ and $y = 0$ in equation .... (III)

$\therefore\ 0.e^{-0} = -2(0)e^{-0} - 2e^{-0} + c$

$\therefore\ 0 = -2 + c$

$\therefore\ 2 = c$ Put in (III)

$y.e^{-x} = -2x\ e^{-x} - 2e^{-x} + 2$

$\therefore\ y = -2x - 2 + 2e^{x}$

$\therefore\ 2x + y + 2 = 2e^{x}$ is the solution.

Ex. 9) The population of a town increasing at a rate proportional to the population at that time. If the population increases from 40 thousands to 60 thousands in 40 years, what will be the population in another 20 years? (Given $\sqrt{\frac{3}{2}} = 1.2247$).

Solution : Let P be the population at time t. Since rate of increase of P is a proportional to P itself, we have,

$\frac{dp}{dt} = k \cdot P$ .... (1)

Where k is constant of proportionality.

Solving this differential equation, we get

$P = a \cdot e^{kt}$, where $a = e^{c}$ .... (2)

Initially $P = 40{,}000$ when $t = 0$

$\therefore$ From equation (2), we have

$40{,}000 = a \cdot 1 \quad \therefore\ a = 40{,}000$

$\therefore$ Equation (2) becomes

$P = 40{,}000 \cdot e\ kt$ .... (3)

Again given that $P = 60{,}000$ when $t = 40$

$\therefore$ From equation (3), $60{,}000 = 40{,}000 \cdot e^{40k}$

$\therefore\ e^{40k} = \frac{3}{2}$ .... (4)

Now we have to find P when $t = 40 + 20 = 60$ years.

$\therefore$ From equation (3), we have

$P = 40{,}000 \cdot e^{60k} = 40{,}000\,(e^{40k})^{\frac{3}{2}}$

$= 40{,}000 \left(\frac{3}{2}\right)^{\frac{3}{2}}$

$= 40{,}000 \left(\frac{3}{2}\right) \sqrt{\frac{3}{2}}$

$= 40{,}000 \left(\frac{3}{2}\right) (1.2247) = 73{,}482$

$\therefore$ Required population will be 73,482.

Problems For Practice

Q.1 A) Select and write the most appropriate answer from the given alternatives for each sub-question :

1) The order and degree of $\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}} = 8\frac{d^3y}{dx^3}$ are respectively

a) 3,1 b) 1,3 c) 3,3 d) 1,1

2) The differential equation of $y = k_1 e^x + k_2 e^{(-x)}$ is

a) $\frac{d^2y}{dx^2} - y = 0$ b) $\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0$

c) $\frac{d^2y}{dx^2} + y = e^{(-x)}$ d) $\frac{d^2y}{dx^2} + y = e^x$

3) The integrating factor of $\frac{dy}{dx} + y = e^{(-x)}$ is

a) x b) $-x$ c) e^x d) $e^{(-x)}$

4) The solution of $x\frac{dy}{dx} = y \log y$ is

a) $y = ae^x$ b) $y = ae^{2x}$ c) $y = ae^{(-2x)}$ d) $y = e^{ax}$

5) Bacterial increases at the rate proportional to the number present. If the original number M doubles in 3 hours, then the number of bacteria will be 4M in

a) 4 hours b) 6 hours c) 8 hours d) 10 hours

B)State whether the following statements are True or False :

1) The degree of the differential equation $e^{\frac{dy}{dx}} = \frac{dy}{dx} + c$ is not defined.

2) The integrating factor of the differential equation $\frac{dy}{dx} - y = x$ is e^x.

3) The number of arbitrary constants in the particular solution of a differential equation of third order is 0.

C)Fill in the following blanks :

1) The order of highest derivative occurring in the differential equation is called .... of the differential equation.

2) The power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any is called .... of the differential equation.

3) A solution of differential equation which can be obtained from the general solution by giving particular values to the arbitrary constants is called .... solution.

4) Order and degree of a differential equation are always .... integers.

5) The differential equation by eliminating arbitrary constants from bx + ay = ab is ....

Q. 2) Obtain the differential equation by eliminating arbitrary constants form the following.

a) $y = c_2 + \frac{c_1}{x}$ b) $y = c_1 e^{2x} + c_2 e^{5x}$

Q. 3) Form the differential equation whose general solution is $x^3 + y^3 = 35ax$.

Q. 4) Verify that : $xy = \log y + c$ is a solution of the differential equation $\frac{dy}{dx} = \frac{y^2}{1 - xy}$.

Q. 5) For the following differential equation, find the particular solution satisfying the given condition. $(x-y^2x)\,dx - (y+x^2y)\,dy = 0$ when $x=2$, $y=0$.

Q. 6) Solve the following differential equations.

1) $\frac{dy}{dx} = 1 + x + y + xy$

2) $xy\frac{dy}{dx} = x^2 + 2y^2$

3) $\left(1 + 2e^{\frac{x}{y}}\right) dx + 2e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) dy = 0$

4) $\frac{dy}{dx} + \frac{2}{x} y = x^2$

Q. 7) The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number double in 1 hour, find the number of bacteria after 5/2 hour. (Take $\sqrt{2}$ = 1.414)

Q. 8) The rate of disintegration of a radioactive element at time t is proportional to its mass at that time. The original mass of 800 gm will disintegrate into its mass of 400 gm after 5 days. Find the mass remaining after 30 days.

Q. 9) Solve the differential equation $\frac{dy}{dx} + y = 3$.

Solution : The equation $\frac{dy}{dx} + y = 3$ is of the form $\frac{dy}{dx} + Py = Q$

Where P = 1 and Q = 3

Integrating factor. = $e^{\int p dx}$ = $\boxed{}$

The solution of the linear differential equation is

$y \boxed{} = \int$ Q(I.F.) dx + c.

$y\, e^x = 3\int \boxed{}\, dx + c.$

$\boxed{}\; y\, e^x = 3 \boxed{}$ + c. This is the general solution.

Q. 10) The rate of depreciation $\frac{dv}{dt}$ of a machine is inversely proportional to the square of t + 1, where V is the value of the machine t years after it was purchased. The initial value of the machine was Rs. 8,00,000 and its value decreased Rs. 1,00,000 in the first year. Find its value after 6 years.

Solution : According to the given condition,

$$\frac{dv}{dt} \alpha \frac{1}{(t+1)^2}$$

$$\frac{dv}{dt} = \frac{k}{(t+1)^2}$$ .... (Negative sign indicate disintegration)

Where k is constant of proportionality.

$$\therefore\ dv = \frac{-k}{(t+1)^2}\, dt$$

Integrating both sides we get, $\int 1 dv = \int \frac{-k}{(t+1)^2}\, dt$

$$v = \frac{k}{\boxed{}} + c \text{ .... (i)}$$

When t = 0, v = 8,00,000

8,00,000 = k + c .... (ii)

When t = 1, v = 7,00,000

$$7{,}00{,}000 = \frac{k}{2} + c \text{ .... (iii)}$$

Solving equation (ii) and (iii), we get

$k = \boxed{}$ and $c = \boxed{}$

When t = 6, we get $v = \dfrac{k}{7} + c$

∴ The Value of machine after 6 years $\boxed{}$

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Answer Key

1. Mathematical Logic

1) i) c, ii) c, iii) d, iv) d, v) b, vi) b, vii) b, viii) d, ix) b, x) a,
xi) c, xii) a, xiii) d, xiv) c xv) a, xvi) a, xvii) d

2) i) F, ii) T, iii) T, iv) F, v) F, vi) T, vii) T, viii) F, ix) T, x) F,
xi) T

3) i) T,F, ii) converse iii) True, iv) all complex numbers are not integer,
v) $(p \wedge c) \vee (\sim q \vee t)$

4) i) If I finish my work then I shall come.
ii) If the duties are performed sincerely then the rights follow.
iii) If $x = 1$ then $x^2 = x$.
iv) If a man is rational, then he is rich.
v) If I purchase a car then I get bonus.

5) i)

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	F
F	F	T	T	F	F	T

ii) Since truth values in last column are true and false both therefore it is contingency.

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim (p \wedge \sim q)$	$\sim (p \wedge \sim q) \vee \sim p$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	F	T	T

iii)

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \vee r$	$\sim r \vee (p \wedge q)$	$[(p \wedge q) \vee r] \wedge [\sim r \vee (p \wedge q)]$
T	T	T	F	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	F
T	F	F	T	F	F	T	F
F	T	T	F	F	T	F	F
F	T	F	T	F	F	T	F
F	F	T	F	F	F	F	F
F	F	F	T	F	F	T	F

iv)

p	q	r	$\sim r$	$p \wedge q$	$q \rightarrow r$	$(p \wedge q) \wedge (q \rightarrow r)$	$p \rightarrow r$	$[(p \wedge q) \wedge (q \rightarrow r) \leftrightarrow (p \rightarrow r)$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	F	F	T
T	F	T	F	F	T	F	T	F
T	F	F	T	F	T	F	F	T
F	T	T	F	F	T	F	T	F
F	T	F	T	F	F	F	T	F
F	F	T	F	F	T	F	T	F
F	F	F	T	F	T	F	T	F

Since truth values in last column are true and false both therefore it is contingency.

i) Converse : If the measure of an angle is not 90°, then it is a right angle.

Inverse : If an angle is not a right angle, then its measure is not 90°

Contra positive : If the measure of an angle is not 90° then it is not a right angle.

ii) Converse : If areas of two triangles are equal then they are congruent.

Inverse : If two triangles are not congruent then their areas are not equal.

Contra positive : If areas of two triangles are not equal then they are not congruent.

iii) Converse : If $f(x)$ is divisible by (x–2) then $f(2) = 0$.

Inverse : If $f(2) \neq 0$ then $f(x)$ is not divisible by $(x–2)$

Contra positive : If $f(x)$ is not divisible by $(x–2)$ then $f(2) \neq 0$.

7) i) Some equilateral triangles are not isosceles.

ii) All complex numbers are real.

iii) Some students have not paid the fees.

iv) $\exists\, n \in N$, such that $n - 8 \le 9$

v) $\forall\, x \in R\, ,\, x^2 \le x$.

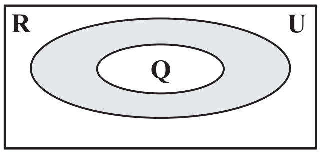
vi) The leaders are not corrupt but democracy does not survive.

vii) A person is successful but he is not honest or a person is honest but he is not successful.

8) i) U : Set of complex numbers

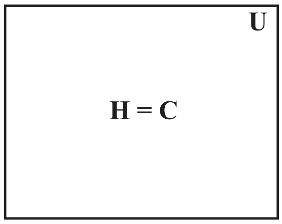
R : Set of all real numbers.

Q : Set of all rational numbers.



ii) U : Set of all human beings.

E : Set of all students.

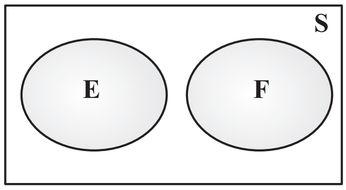


iii) U : Human being

H : Happy people

C : People whose Concepts of math are clear

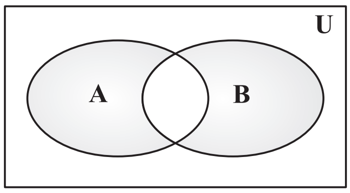
F : Set of all lazy people.



iv) U : Human being

A : Commerce students

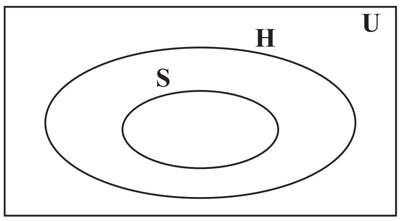
B : Students taken maths.



ix) U : All calendar days.

S : Sundays

H : Holidays



2. Matrices

Q.I		Q.II		Q.III	
1)	b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	1)	False	1)	$\begin{bmatrix} -8 & -5 \\ -2 & 1 \end{bmatrix}$
2)	b) Symmetric Matrix	2)	False	2)	$\begin{bmatrix} 6 & -12 & 9 \\ -2 & -8 & 6 \\ 2 & -4 & 3 \end{bmatrix}$
3)	b) 7	3)	False	3)	$x = 2, y = 3$
4)	a) 2	4)	True	4)	$k = \frac{-6}{7}$
5)	d) $\begin{bmatrix} 2 & 3 \\ -6 & 4 \end{bmatrix}$	5)	True	5)	$\begin{bmatrix} 3 & -3 & 0 \\ 4 & 3 & 5 \end{bmatrix}$
6)	a) 6, 4	6)	True		0
7)	c) ± 5	7)	False		$\begin{bmatrix} 2 & 2 \\ 3 & -4 \end{bmatrix}$
8)	c) $\begin{bmatrix} 2 & -1 \\ -11 & 6 \end{bmatrix}$	8)	False		$\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$
Q.IV		Q.V		Q.VI	
1)	$x = \begin{bmatrix} -1 & \frac{2}{5} \\ \frac{6}{5} & \frac{19}{5} \\ \frac{19}{5} & \frac{26}{5} \end{bmatrix}$		$x = 1, y = 5, z = 5$		$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, 5, $\frac{1}{9}\begin{bmatrix} 7 & 4 & -1 \\ 5 & -1 & -2 \\ 3 & 3 & -3 \end{bmatrix}$
2)	$A = \begin{bmatrix} 3 & \frac{-14}{3} & \frac{-8}{3} \\ -2 & 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 0 & \frac{-10}{3} & \frac{-16}{3} \\ 0 & 0 & 5 \end{bmatrix}$		$x = \frac{26}{7}, y = \frac{30}{7}$	2)	$x - 2y + z$ $\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & 0 \end{bmatrix}$ (1, 2, 3)

Q.IV		Q.V		Q.VI	
	$A^{-1} = \frac{1}{40}\begin{bmatrix} 19 & 5 & -27 \\ -2 & 10 & -14 \\ -3 & -5 & 19 \end{bmatrix}$	8)	$x = 1, y = 5$		
			$k = 1$		

3. Differentiation

1) $(3x-1)(3x^2-2x-1)^{\frac{-3}{2}}$

2) $\frac{5}{3}(6x+8)(3x^2+8x-6)^{\frac{2}{3}}$

3) $\frac{1}{(1+75x^2)}$

4) $\frac{(2x^2+5)^2}{(-6x^2-28x+15)}$

5) $\frac{y}{2\sqrt{x}}(\log x+2)$

6) $x^x(1+\log x)+10^x\log 10$

7) $\frac{y}{x}$

8) $\frac{x\log 2+y}{x\log^{2x}}$

9) 2m

10) $\frac{x}{y}$

II) Fill in the blanks :

1) $9(5x^3-4x^2-8x)8(15x^2-8^x-8)$

2) $a^{(1+\log x)}\log a.\frac{1}{x}$

3) $\frac{dy}{dx}=\frac{1}{(-x^2e-x+2xe^{-x}+2)}$

4) $\frac{e^x}{1-x}$

5) $10x^9$

6) $2e^{(2x+5)}$

7) $-\sqrt{\frac{x}{y}}$

8) $x.5^x\log 5$

9) $x.e^x$

10) 2

III) State whether each of the following is True or False.

1	2	3	4	5	6	7	8	9	10	11
False	True	True	False	True	False	False	False	False	True	True

IV) Solve the following :

1) $(\log x)^x \left[\dfrac{1}{\log x} + \log(\log x) \right]$

2) $\dfrac{4}{5}(3x^2 + 8x + 5)^{\frac{-1}{5}}(6x + 8)$

3) $(7x-1)^x \left[\log(7x-1) + \dfrac{7x}{7x-1} \right]$

4) $\dfrac{dy}{dx} = \dfrac{(2x^2+5)^2}{(-6x^2-28x+15)}$

4. Application of Derivatives

A) 1) F, 2. T, 3. F, 4. F, 5.T

B) 1) $6x + 6$ 2) Relatively inelastic 3) $3x^2 + 10^x$ 4) $-1/5$ 5) 0.2479

C) 1-B) $-2/3$ 2-B) $2 < x < 3$ 3-A) 3/2 4-C) 6.5 5-B) 40

1) tangent $x + y = 2$ normal $x - y = 0$

2) decreasing for all x in R, $x \neq 0$

3) $-1 < x < 5$

4) 10 and 10

5) D = 30

6) i) R max $x = 20$ ii) C min $x = 0$

7) $x < 27$, $x < 20$

8) 3.6

9) MR = 15

10) MPC = 0.485, MPS = 0.515, APC = 0.285, APS = 0.715

1) $f'(x) = \dfrac{dy}{dx}\left(x + \dfrac{1}{x}\right) = 1 - \dfrac{1}{-x^2}$

f is decreasing if $f'(x) < 0$

ie if $1 - \dfrac{1}{x^2} < 0$

ie if $x^2 < 1$

ie $-1 < x < 1$

so f is decreasing for $x \in (-1, 1)$

2) MPC = 0.675, MPS = 0.325, APC = 0.375, APS = 0.625

5. Integration

I] MCQ .

1) $C - \frac{1}{4}$ 2) A $-$ log x $-$ log $(1 - x) + c$ 3) A $- e^x - \frac{1}{3e^{3x}} + c$

II] Fill in the blanks .

1) $4 \log |x - 1|$ 2) 2 3) $\frac{1}{3}$

III] True or false

1)True 2)False 3)True

Three marks examples :

1) $\frac{-1}{18(6x+5)^3} - \frac{-1}{24(8-3x)^8} + c$

2) $\frac{1}{3}\left[x^{\frac{3}{2}} - (x-2)^{\frac{3}{2}}\right]$

3) $x^4 - 4x^3 + \frac{9}{2}x^2 + c$

4) Log $|\log(\log x)| + c$

5) Log $|e^x + 1| + c$

6) $2\sqrt{1+x^3} + c$

7) $\frac{-1}{2}x + 2\log|2e^x - 8| + c$

8) $\frac{-1}{12} \log \left| \frac{3x-2}{3x+2} \right| + c$

9) $\frac{1}{2} \log \sqrt{x^2 - \left|\left(\frac{3}{2}\right)^2\right|} + c$

10) $x(\log x - 1) + c$

Four marks :

1) $\log|\log x + \sqrt{(\log x)^2 - \sqrt{5^2}} + c$

2) $\frac{1}{3}\log\left|\frac{2x-1}{2x+2}\right|+c$

2) $\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+c$

4) $\log|(x-4)\sqrt{x^2-8x-20}\,|+c$

5) $\frac{x^4\log x}{4}\frac{x^4}{16}+c$

6) $\frac{1}{3}x^2e^{3x}-\frac{2}{9}x\,e^{3x}+\frac{2}{27}e^{3x}+c$

7) $\frac{(x-2)}{2}\sqrt{x^2-4x-5}-\frac{9}{2}\log|(x-2)+\sqrt{x^2-4x-5}\,|+c$

6. Definite Integration

1) $\frac{8}{3}\log\frac{35}{8}$ 2) $\frac{32}{5}$ 3) $\sqrt[4]{2}$ 4) $\frac{5}{2}$ 5) $\frac{3}{2}$

6) $\frac{1}{42}$ 7) $\log 27-2$ 8) $\frac{1}{2}\log\left(\frac{6}{5}\right)$ 9) $\frac{1}{2}\log(6)$ 10) $\frac{1}{2}\log\frac{x+5\sqrt{3}}{+1\sqrt{3}}$

7. Application of Definite Integration

I. 1) (A) $\frac{26}{3}$ sq. units 2) (D) $\frac{4}{3}$ sq. units

II. 1) 25 sq. units 2) 16 sq. units

III. 1) True 2) False

IV. 1) (i) $\frac{62}{3}$ sq. units (ii) 18 sq. units (iii) $\frac{16}{3}$ sq. units (iv) $\frac{64}{3}$ sq. units

2) $\frac{50}{3}\sqrt{5}$ sq. units 3) $\frac{23}{3}\sqrt{5}$ sq. units

V. (9,9) , $\frac{x^2}{9}$, $\frac{2}{3}x^{\frac{2}{3}}$, 27

8. Differential Equation

Q. 1 (A) i) C ii) A iii) C iv) D v) B

(B) i) True ii) False iii) True

(C) i) Order ii) Degree iii) Particular iv) Positive v) $\frac{d^2y}{dx^2} = 0$

Q. 2 a) $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0.$ b) $\frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 10y = 0$

Q. 3 $2x^3 - y^3 + 3xy^2 + \frac{dy}{dx} = 0$

Q.4 $(1 + x^2)(1 - y^2) = 0$

Q.5 i) $\log |1+ y| = x + \frac{x^2}{2} + c$ (ii) $x^2 + y^2 = cx^4$

iii) $x + 2ye^{\frac{x}{y}} = c$ iv) $5x^2y = x^5 + c....[c = 5c_1]$

Q.7 5656

Q.8 12.5 mg

■■■

Std. XII - Subject : Mathematics and Statistics
(Commerce) Part - I

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