"Comprehensive Support for Students in Mathematics subject seeking to Overcome Past Setbacks."

MATHEMATICS AND STATISTICS

Std. - XII

(Arts and Science)

Part - I

State Council of Educational Research and Training, Maharashtra, Pune

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'Comprehensive Support for Students in Mathematics subject seeking to Overcome Past Setbacks.'

Specialized Mathematics Study Materials for HSC Students

Subject : Mathematics and statistics (Arts and Science) Code : 40

OBJECTIVES OF THE BOOKLET

This booklet is prepared for the help of the students who will be appearing for the Supplementary Examination and thereafter too. It is prepared as such students could not score the minimum score to pass in the written Board examination held in February 2024.

This booklet is designed to boost the confidence of the students. It will definitely help them to score good marks in the forthcoming examination. It will be a great support for the students who lack behind others.

It is prepared in a systematic and easiest way by the expert teachers. The students are aware of the text book as well as the examination pattern (MCQ's, 1 Mark, 2 Marks, 3 Marks and 4 Marks questions). Still, this booklet elaborates every segment in detail. It considers the level of the students.

By studying as suggested in the booklet, we are quite sure that the students will be able to practice a lot with given guidelines. They will score and step into the world of success.

The main objectives can be summarized as under :

1) To facilitate the essential study material to the students to confidently face the HSC Board Examination.
2) To help every low achiever student to achieve 100% success at the HSC Board Examination.
3) To motivate the students to score more than their expectation in the Mathematics Subject which they find as most difficult.
4) To include tools and exercises that allow students to evaluate their own progress and understand their improvement areas.
5) To help the teachers to reach out to students who struggle to pass in the Mathematics subject at the HSC Board Exam with the help of this material.
6) Each chapter in the booklet contains important concepts in short.
7) Based on these concepts simple solved examples are given.
8) Practice questions with hints and answers are given.
9) Two practice question papers will definitely help students.

INTRODUCTION

Dear Students,

It does not matter if you did not score well in the regular examination held in February 2024. Remember, "every setback is a setup for a comeback." Your previous attempt must have taught you something valuable. We believe in your potential to overcome this hurdle and excel in your upcoming exams.

After a comprehensive analysis of the results, SCERT, Maharashtra, Pune has taken an initiative for the upliftment of students who could not achieve the minimum passing score. It was found that some fundamental concepts were not clear to the students. Hence, a significant effort was made to prepare this booklet.

This booklet is designed specifically for those who did not achieve the desired results in their previous Mathematics exam. We understand that facing a setback can be challenging, but it also presents an invaluable opportunity for growth and learning. Our goal with this booklet is to provide you with comprehensive resources and targeted exercises to help you strengthen your understanding of key mathematical concepts. We have carefully curated the content to address common areas of difficulty and to reinforce fundamental principles essential for success in Mathematics.

This booklet will help you to prepare for the supplementary examination. Through a combination of clear explanations, step-by-step problem-solving strategies, and ample practice questions, we aim to build your confidence and competence in the subject. Remember, perseverance and a positive mindset are crucial as you work through this material.

Use this booklet diligently, seek help when needed, and stay committed to your studies. With dedication and effort, you can turn this experience into a stepping stone toward academic success. This resource will also prove to be extremely useful for teachers as they assist students in preparing for the supplementary examination. It will boost your confidence to appear for the exam once again. New students in the coming years can also benefit from this booklet. Best wishes for your journey ahead.

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Part I

1. Mathematical Logic

1.1 Statement :

A statement is a declarative (assertive) sentence which is either true or false, but not both simultaneously. Statements are denoted by $p, q, r,$ .....

Truth value of a statement :

Each statement is either true or false. If a statement is true then its truth value is 'T' and if the statement is false then its truth value is F.

Following sentences are statements.

i) $5 \times 2 = 11$

ii) Every triangle has three sides.

iii) Mumbai is the capital of Maharashtra.

Following sentences are not statements.

i) Please, give your Pen.

ii) What is your name?

iii) Open the window.

Note : Interrogative, exclamatory, command, order, request, suggestion are not statements.

Let us learn these statements :

i) For $x = 6$ it is true but for other than 6 it is not true. Here, we cannot determine the truth value.

ii) For 'It is black in colour.' the truth value varies from person to person. In the above sentences, the truth value depends upon the situation. Such sentences are called as open sentences. Open sentence is not a statement.

Logical connectives, simple and compound statements :

The words or phrases which are used to connect two statements are called logical connectives.

We will study the connectives 'and', 'or', 'if ..... then', 'if and only if ', 'not".

Simple and Compound Statements : A statement which cannot be split further into two or more statements is called a simple statement. If a statement is the combination of two or more simple statements, then it is called a compound statement.

"3 is a prime and 4 is an even number", is a compound statement.

"3 and 5 are twin primes", is a simple statement.

1.2 Statement Pattern, Logical Equivalence, Tautology, Contradiction, Contingency.

1) **Statement Pattern :** Letters used to denote statements are called statement letters. Proper combination of statement letters and connectives is called a statement pattern. Statement pattern is also called as a proposition. $p \rightarrow q$, $p \wedge q$, $\sim p \vee q$ are statement patterns. p and q are their prime components.

   A table which shows the possible truth values of a statement pattern obtained by considering all possible combinations of truth values of its prime components is called the truth table of the statement pattern.

2) **Logical Equivalence :**
   Two statement patterns are said to be equivalent if their truth tables are identical. If statement patterns A and B are equivalent, we write it as A ≡ B.

3) **Tautology, Contradiction and Contingency :**
   - **Tautology :** A statement pattern whose truth value is true for all possible combinations of truth values of its prime components is called a tautology. We denote tautology by t.

     Ex. Statement pattern $p \vee \sim p$ is a tautology.
   - **Contradiction :** A statement pattern whose truth value is false for all possible combinations of truth values of its prime components is called a contradiction. We denote contradiction by c.

     Ex. Statement pattern $p \vee \sim p$ is a contradiction.
   - **Contingency :** A statement pattern which is neither a tautology nor a contradiction is called a contingency.

     Ex. $p \wedge q$ is a contingency.

Ex. 1) $[(p \vee q) \vee r] \leftrightarrow (p \vee (q \vee r)]$

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$	$[(p \vee q) \vee r] \leftrightarrow [p \vee (q \vee r)]$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	T

All the truth values in the last column are T, hence it is tautology.

Ex. 2) $[p \wedge p \rightarrow \sim q)] \rightarrow q$

p	q	$\sim q$	$p \rightarrow \sim q$	$p \vee (p \rightarrow \sim q$	$p \vee (p \rightarrow \sim q) \rightarrow q$
T	T	F	F	F	T
T	F	T	T	T	F
F	T	F	T	F	T
F	F	T	T	F	T

Truth values in the last column are not identical. Hence it is contingency.

Ex. 3) $(p \wedge q) \wedge (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \vee \sim q$	$(p \wedge q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

All the truth values in the last column are F. Hence it is contradiction.

Ex. 4) Prove that i) $p \rightarrow q \equiv \sim p \vee q$ **ii)** $p \rightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

I	II	III	IV	V	VI	VII	VIII
p	q	$\sim p$	$p \rightarrow q$	$q \rightarrow p$	$p \rightarrow q$	$\sim p \rightarrow q$	$p \rightarrow q \wedge q \rightarrow p$
T	T	F	T	T	T	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	T	F
F	F	T	T	T	T	T	T

Columns (IV, VII) and (VI, VIII) are identical.

$\therefore$ $p \rightarrow q \equiv \sim p \vee q$ and $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ are proved.

1.3 Quantifiers, Quantified Statements, Duals, Negation of Compound Statements, Converse, Inverse And Contra positive of Implication.

Quantifiers : A sentence that contains one or more variables is called an open sentence. An open sentence becomes true or false statement when we replace the variables by some specific values form a given set. The phrases that quantify the variables in open sentences are called quantifiers.

There are two types of quantifiers.

(i) Universal quantifiers : The quantifiers 'for all' or 'for every' is called universal quantifiers and is denoted by $\forall$.

(ii) Existential quantifier : The quantifiers' for some' or' for one' or 'for each' or 'there exist at least one' or simply' there exists' is called existential quantifier and is denoted by $\exists$.

Quantified statement : An open sentence with a quantifier becomes a statement in logic. Such statement is called a quantified statement. i.e. statements involving quantifiers are called quantified statements.

Remarks :

(i) Every quantified statement corresponds to a collection and a condition.

(ii) A statements quantified by universal quantifier for all '$\forall$' is true if all objects in the collection satisfy the condition and is false if at least one object in the collection does not satisfy the condition.

(iii) A statement quantified by existential quantifiers '$\exists$' is true if at least one object in the collection satisfies the condition and is false if no object in the collection satisfies the condition.

Ex. 1) If A = {1, 2, 3, 4, 5, 6, 7}, determine the truth value of the following.

i) $\exists\ x \in A$ such that $x - 4 = 3$ ii) $\forall\ x \in A\ ,\quad x + 1 > 3$

Solution : i) For $x = 7, x - 4 = 7 - 4 = 3$

$\therefore\ x = 7$ satisfies the equation $x - 7 = 3$

$\therefore$ The given statement is true and its truth value is T.

ii) For $x = 1, x + 1 = 1 + 1 = 2$ which is not greater than or equal to 3.

$\therefore$ For $x = 1\ x^2 + 1 > 3$ is not true.

$\therefore$ The truth value of given statement is F.

Duals : Two compound statements S_1 and S_2, are said to be duals of each other if any one of them can be obtained from the other by replacing conjunction ($\wedge$) by disjunction ($\vee$) and vice versa. The connectives $\wedge$ and $\vee$ are called duals of each other. e.g. The duals of $p \vee q$ is $p \wedge q$.

Remarks :

(i) If a compound statements 'S' contains tautology then is denoted by 't' and if it contains contradiction then it is denoted by 'f'.

(ii) Two statements contain logical connectives like $\wedge$, $\vee$ and letters 't' and 'c' then they are said to be dual of each other if one of them is obtained from other by interchanging with $\wedge$ and $\vee$ and 't' with 'c'. eg. The dual of ‘t $\wedge$ p' is ‘c $\vee$ p’ and the dual of ‘t $\vee$ p' is ‘c $\wedge$ p’

(iii) The symbol negation '~' is not changed while finding dual.

(iv) The special statement t and c are called dual of each other.

(v) The results $\sim (p \wedge q) \equiv \sim p \vee \sim q$ and $\sim (p \vee q) \equiv \sim p \wedge \sim q$ are known as 'De-morgan's laws'. These results are called duals of each other.

Principle of Duality :

- If a compound statement S_1 contains only $\sim$, $\wedge$, $\vee$ and statement S_2 is obtained from S_1 by replacing $\wedge$ by $\vee$ and $\wedge$ by $\wedge$ then S_1 is a tautology if S_2 is a contradiction.

Ex. 1) Write the duals of each of the following :

i) $p \wedge [\sim q \vee (p \wedge q) \vee \sim r]$ ii) $(p \vee q) \wedge t$

iii) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ iv) $(p \wedge t) \vee (c \wedge \sim q)$

Solution : i) $p \vee [(\sim q \wedge (p \vee q) \wedge \sim r]$ ii) $(p \wedge q) \vee c$

iii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ iv) $(p \vee c) \wedge (t \vee \sim q)$

Negation of compound statements :

The negation of simple statements is obtained by inserting 'not' at the appropriate place in the statements.

The following rules are considered to form the negation of a compound statement.

(1) Negation of Negation : The negation of negation of a simple statement is the statement itself. If p is a simple statements then $\sim (\sim p) \equiv p$.

(2) Negation of conjunction : The negation of $p \wedge q$ is $\sim p \vee \sim q$ i.e. $\sim (p \wedge q) \equiv \sim p \vee \sim q$

The negation of "6 is even and perfect number" is "6 not even or not perfect number".

(3) Negation of disjunction : The negation of $p \vee q$ is $\sim p \wedge \sim q$ i.e. $\sim (p \vee q) \equiv \sim p \wedge \sim q$

The negation of "x is prime or y is even" is "x is not prime and y is not even".

(4) Negation of implication : The conditional statement "If p then q" is false only in the case "p is true and q is false". In all other cases it is true. The negation of the statement, "If p then q" is the statement "p and not q". i.e. $\sim (p \to q) \equiv p \wedge \sim q$

(5) Negation of biconditional : The negation of the statement "p if and only if q" is the statement "p and not q, or q and not p".

i.e. $\sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

(6) Negation of quantified statement : The negation of a quantified statement is obtained by replacing the word 'all' by 'some', 'for every by 'there exists' and vice versa.

Ex. Write the negations of the following.

i) $3 + 3 < 5$ or $5 + 5 = 9$

Solution : Let $p : 3 + 3 < 5 : q : 5 + 5 = 9$

Given statement is $p \vee q$ and its negation is $\sim(p \vee q)$ and $\sim(p \vee q) \equiv \sim p \wedge \sim q$

$\therefore$ The negation of given statement is $3 + 3 \geq 5$ and $5 + 5 \neq 9$

ii) 7 > 3 and 4 > 11

Solution : Let p : 7 > 3 ; q : 4 > 11 The given statement is $p \wedge q$

Its negation is $\sim (p \wedge q)$ and $\sim (p \wedge q) \equiv \sim p \vee \sim q$

$\therefore$ The negation of given statement is $7 \leq 3$ or $4 \leq 11$

iii) The number is neither odd nor perfect square.

Solution : Let p : The number is odd

q : The number is perfect square.

Given statement can be written as 'the number is not odd and not perfect square'

Given statement is $\sim p \wedge \sim q$

Its negation is $\sim (\sim p \wedge \sim q) \equiv p \vee q$

The negation of given statement is 'The number is odd or perfect square'

iv) The number is an even number if and only if it is divisible by 2.

Solution : Let p : The number is an even number.

q : The number is divisible by 2.

Given statement is $p \leftrightarrow q$

Its negation is $\sim (p \leftrightarrow q)$

But $\sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

$\therefore$ The negation of given statement is :

'A number is even but not divisible by 2 or a number is divisible by 2 but not even'.

v) All natural numbers are rational.

Solution : Some natural numbers are not rational.

vi) Some students of class X are sixteen year old.

Solution : No student of class X is sixteen year old.

vii) $\exists$ n $\in$ N such that n + 8 > 11

Solution : $\forall$ n $\in$ N, n + 8 $\leq$ 11

viii) $\forall$ x $\in$ N, $2x + 1$ is odd

Solution : $\exists \times \in$ N such hat $2x + 1$ is not odd.

Converse, Inverse and contra positive :

From implication $p \to q$ we can obtain three implications, called converse, inverse and contra positive.

i) $q \to p$ is called the converse of $p \to q$

ii) $\sim p \to \sim q$ is called the inverse of p $\to$ Ex.

iii) $\sim q \rightarrow \sim p$ is called the contra positive of $p \rightarrow$ Ex.

Ex. 1) Write the converse, inverse and contra positive of the following statement

'If a function is differentiable then it is continuous.'

Solution : Let p : A function is differentiable

q : A function is continuous.

$\therefore$ Given statement is $p \rightarrow q$

i) Its converse is $q \rightarrow p$

If a function is continuous then it is differentiable.

ii) Its inverse is $\sim p \rightarrow \sim q$

If a function not differentiable then it is not continuous.

iii) Its contra positive is $\sim q \rightarrow \sim p$

If a function is not continuous then it is not differentiable.

Exercise 1.1

1) Write down the following statements in symbolic form.

i) A triangle is equilateral if and only if it is equiangular.

ii) Price increases and demand falls.

2) Write the truth values of the following statements.

i) Two is the only even prime number

ii) $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, for all $\theta \in R$

iii) $\exists\ n \in N$ such that $n + 5 > 10$

3) Write the dual of the following statements.

i) $(p \vee q) \wedge t$

ii) Madhuri has curly hair and brown eyes.

iii) $\sim p \wedge (q \vee c)$

iv) 'Shweta is a doctor or Seema is a teacher.'

v) $p \wedge \sim p \equiv c$

Exercise 1.2

1) Write the truth values of the following statements.

i) 2 is a rational number and $\sqrt{2}$ is an irrational number.

ii) $2 + 3 = 5$ or $\sqrt{2} + \sqrt{3} = \sqrt{5}$

iii) $\sqrt{5}$ is an irrational number but $3 + \sqrt{5}$ is a complex number.

iv) If the statements p, q are true statements and r, s are false then determine the truth value of $(p \rightarrow q) \vee (r \rightarrow s)$

v) If p, q, r are the statements with truth values T, F, T respectively then find the the truth value of $(r \wedge q) \leftrightarrow \sim p$

vi) $\forall n \in N$, $n^2 + n$ is an even number and $n^2 - n$ is an odd number.

2) Write the negations of the following statements.

i) $\forall n \in N, n + 7 > 6$

ii) The kitchen is neat and tidy

iii) All students of this college live in the hostel

iv) 6 is an even number or 36 is a perfect square

v) If diagonals of a parallelogram are perpendicular, then it is a rhombus.

vi) Mangos are delicious, but expensive.

vii)A person is rich if and only if he is a software engineer

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Exercise 1.3

1) Using truth table prove that :

i) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

ii) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

iii) $\sim p \wedge q \equiv \sim(p \rightarrow \sim q)$

iv) $\sim p \wedge q \equiv (p \vee q) \wedge \sim p$

v) $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

vi) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

viii) $(p \wedge q) \vee \sim q \equiv p \vee \sim q$

2) Examine whether the following statement pattern is tautology, contradiction or contingency.

i) $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$

ii) $(p \wedge \sim q) \leftrightarrow (p \rightarrow q)$

iii) $(\sim p \wedge \sim q) \vee q$

iv) $(p \wedge q) \vee (p \wedge r)$

v) $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$

vi) $[(p \rightarrow q) \wedge q] \rightarrow p$

3) Write the converse, inverse and contra positive of the following statements :

i) "If two triangles are congruent then their areas are equal."

ii) "If it rains then the match will be cancelled."

iii) "If an angle is right angle then its measure is 90°"

iv) "If a sequence is bounded, then it is convergent."

v) "If $x < y$ then $x^2 < y^2$"

2. Matrices

2.1 Elementary Transformation : Two types of elementary transformation of a matrix.

1) Row transformation 2) Column transformation

Elementary Transformation are as follows :

a) Interchange of any two rows or any two columns :

Interchange of ith row and jth row is symbolically denoted by $R_i \leftrightarrow R_j$

Similarly we apply transformation for column also.

For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $R_1 \leftrightarrow R_2$ gives the new matrix $A \sim \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

The Symbol ~ is read as equivalent.

b) Multiplication of the elements of any row or column by a non zero scalar :

If K is non zero scalar and row R_i is multiplied by constant K then Symbolically the transformation is denoted by KR_i or $R_i \to KR_i$

For example if $A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \end{bmatrix}$ then $R_i \to 3R_i$ gives $A \sim \begin{bmatrix} -3 & 6 \\ 5 & 4 \end{bmatrix}$

Similarly we can apply transformation for column also.

C) Adding the scalar multiples of all the elements of any row (column) to corresponding elements of any other row. (column)

If K is non zero scalar and the K multiples of the elements of R_i are added to the corresponding elements of R_j then transformation symbolically denoted as $R_j \to R_j + KR_i$

For example, If $A = \begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix}$ and K = 2 then $R_2 \to R_2 + KR_1$ gives

$$A \sim \begin{bmatrix} 2 & -1 \\ 4 + 2(2) & 5 + 2(-1) \end{bmatrix} \; A \sim \begin{bmatrix} 2 & -1 \\ 8 & 3 \end{bmatrix}$$

We can apply transformation for column also.

Square Matrix : A matrix in which number of rows and number of columns are equal is called Square Matrix. i.e order of the square matrix is $m \times m$, $n \times n$, 2×2, 3×3 etc.

2.2 Inverse of a matrix :

If A is square matrix of order m and $|A| \neq 0$ then A^{-1} is the inverse of A.

we can write $AA^{-1} = \mathrm{I}$ or $A^{-1}A = \mathrm{I}$

Note : Where I is the identity matrix of order m.

Identity Matrix of order 2 and 3 : $I_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Matrix A is invertible if $|A| \neq 0$.

Inverse of non singular matrix of order 2 or 3 by using two methods.

1) **Elementary transformation :**

a) Elementary row transformation.

b) Elementary column transformation.

2) **Adjoint method :**

In this adjoint method, for finding the inverse of a square matrix of order 2 or 3, we find-

i) Minors of each element a_{ij} and it is denoted by M_{ij}

ii) Co-factors of each element a_{ij} and it is denoted by A_{ij} and $A_{ij} = (-1)^{i+j}.\ M_{ij}$

iii) Adjoint matrix of a given matrix is **adjoint of A** = **[*cofactor matrix of A*]^T**

iv) Formula : $A^{-1} = \dfrac{\textbf{\textit{adjoint of A}}}{|A|}$

Application of matrices : System of linear equations solve by using matrix.

1) **Method of Reduction :** Consider AX = B and reduce matrix A into upper triangular matrix by using elementary transformation. Thus the required solution is obtained.

2) **Method of inversion :** consider X = A^{-1}B and Find A^{-1}, by putting A^{-1} and constant matrix B in X = A^{-1}B gives the required solution.

Ex. 1) Check whether the following matrices are invertible or not.

i) A = $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ ii) B = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ iii) C = $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Solution : i) $|A| = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = 4 - 4 = 0$

$\therefore$ *A is singular matrix.*

$\therefore$ *A is not invertible.*

ii) $|B| = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$

$\therefore |B| \neq 0$

$\therefore$ *B is non singular.* $\therefore$ *B is not invertible.*

iii) $|\overline{C}| = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = 1(3-4) - 3(9-2) + 2(6-1)$

$= 1(-1) - 3(7) + 2(5) = -1 - 21 + 10$

$= -12 \neq 0$

$|\overline{C}| \neq 0$

$\therefore$ *C is non singular.*

$\therefore$ *C is invertible.*

Ex. 2) Find the inverse of matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by elementary row transformation.

Solution : As $|\overline{A}| = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2 \neq 0 \quad \therefore A^{-1}$ exists.

Let $A A^{-1} = I$

$\therefore \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By using elementary row transformation,

Using $R_2 - 3R_1$

$\therefore \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

Using $-\frac{1}{2}R_2$

$\therefore \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$

Using $R_1 - 2R_2$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$

$\therefore IA^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} \quad \therefore A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$

Ex. 3) Find the inverse of matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ by using elementary transformation.

Solution : As $|\overline{A}| = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = -1 - 4 = -5 \neq 0$

$\therefore A^{-1}$ exists.

Let $A A^{-1} = I$

$$\therefore \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By using elementary row transformation,

Using $R_2 - 2R_1$

$$\therefore \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

Using $-\frac{1}{5}R_2$

$$\therefore \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

Using $R_1 - 2R_2$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

$$\therefore IA^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

Ex. 4) If $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is non singular matrix then find A^{-1} by elementary row transformation hence write the inverse of $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Solution : Given A is non singular matrix. $\therefore$ A^{-1} exists.

Consider $AA^{-1} = I$

$$\therefore \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $\left(\frac{1}{x}\right) R_1$, $\left(\frac{1}{y}\right) R_2$, $\left(\frac{1}{z}\right) R_3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$$

$$\therefore IA^{-1} = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix} \qquad \therefore A^{-1} = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$$

Hence Inverse of matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Ex. 5) Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ by elementary row transformation.

Solution : As $|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1(16 - 9) - 3(4 - 3) + 3(3 - 4)$

$= 1(7) - 3(1) + 3(-1) = 7 - 3 - 3$

$= 1 \neq 0 \quad \therefore A^{-1}$ exists.

Consider $AA^{-1} = I$

$$\therefore \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $R_2 - R_1$ and $R_3 - R_1$

$$\therefore \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Using $R_1 - 3R_2$

$$\therefore \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 4 & -3 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Using $R_1 - 3R_3$

$$\therefore \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore IA^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \therefore A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Ex. 6) Find cofactors of the elements of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution : Here $a_{11} = 1$, $a_{12} = 2$, $a_{21} = 3$, $a_{22} = 4$

Minors are — cofactors are $A_{ij} = (-1)^{i+j}. M_{ij}$

$M_{11} = Minor\ of\ a_{11} = 4$ — $A_{11} = (-1)^{1+1}.4 = 4$

$M_{12} = Minor\ of\ a_{12} = 3$ — $A_{12} = (-1)^{1+2}.3 = -3$

$M_{21} = Minor\ of\ a_{21} = 2$ — $A_{21} = (-1)^{2+1}.2 = -2$

$M_{22} = Minor\ of\ a_{22} = 1$ — $A_{22} = (-1)^{2+2}.1 = 1$

Ex. 7) Find the adjoint of matrix $A = \begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix}$

Solution : Here $a_{11} = 2$, $a_{12} = -3$, $a_{21} = 3$, $a_{22} = 5$

Minors are — cofactors are $A_{ij} = (-1)^{i+j}. M_{ij}$

$M_{11} = Minor\ of\ a_{11} = 5$ — $A_{11} = (-1)^{1+1}.5 = 5$

$M_{12} = Minor\ of\ a_{12} = 3$ — $A_{12} = (-1)^{1+2}.3 = -3$

$M_{21} = Minor\ of\ a_{21} = -3$ — $A_{21} = (-1)^{2+1}.(-3) = 3$

$M_{22} = Minor\ of\ a_{22} = 2$ — $A_{22} = (-1)^{2+2}.2 = 2$

Cofactor matrix is $\begin{bmatrix} 5 & -3 \\ 3 & 2 \end{bmatrix}$

Adjoint of A = (Cofactor matrix of $A)^T = \begin{bmatrix} 5 & 3 \\ -3 & 2 \end{bmatrix}$

Ex. 8) Find the inverse of matrix $A = \begin{bmatrix} -1 & 5 \\ -3 & 5 \end{bmatrix}$ by adjoint method.

Solution : Here,

Minors are	cofactors are
$a_{11} = -1 M_{11} = 2$	$A_{11} = (-1)^{1+1} M_{11} = 1.(2) = 2$
$a_{12} = 5 M_{12} = -3$	$A_{12} = (-1)^{1+2} M_{12} = (-1)(-3) = 3$
$a_{21} = -3 M_{21} = 5$	$A_{21} = (-1)^{2+1} M_{21} = (-1).5 = -5$
$a_{22} = 2 M_{22} = -1$	$A_{22} = (-1)^{2+2} M_{22} = 1.(-1) = -1$

$\therefore$ Cofactor matrix is $\begin{bmatrix} 2 & 5 \\ -5 & -1 \end{bmatrix}$

$\therefore$ $Adj\ A = (Cofactor\ matrix\ of\ A)^T = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

$\therefore$ $|A| = \begin{bmatrix} -1 & 5 \\ -3 & 1 \end{bmatrix} = -2 - (-15) = -2 + 15 = 13 \neq 0$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{13} & \frac{-5}{13} \\ \frac{3}{13} & \frac{-1}{13} \end{bmatrix}$$

Ex. 9) Find adjoint of matrix $A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

Solution : Here,

Minors are	cofactors are
$a_{11} = 3 M_{11} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = 1$	$A_{11} = (-1)^{1+1} M_{11} = 1$
$a_{12} = 2 M_{12} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = 1$	$A_{12} = (-1)^{1+1} M_{12} = -1$
$a_{13} = 6 M_{13} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 0$	$A_{13} = (-1)^{1+3} M_{13} = 0$
$a_{21} = 1 M_{21} = \begin{bmatrix} 2 & 6 \\ 2 & 5 \end{bmatrix} = -2$	$A_{21} = (-1)^{2+1} M_{21} = -(-2) = 2$
$a_{22} = 1 M_{21} = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix} = 3$	$A_{22} = (-1)^{2+2} M_{22} = 3$
$a_{23} = 2 M_{23} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = 2$	$A_{23} = (-1)^{2+3} M_{23} = -2$
$a_{31} = 2 M_{31} = \begin{bmatrix} 2 & 6 \\ 1 & 2 \end{bmatrix} = -2$	$A_{31} = (-1)^{3+1} M_{31} = -2$

$$a_{32} = 2M_{32} = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} = 0 \qquad A_{32} = (-1)^{3+2} M_{32} = 0$$

$$a_{33} = 5M_{33} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = 1 \qquad A_{33} = (-1)^{3+3} M_{33} = 1$$

Cofactor matrix of A is $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}$

Adjoint of A = $[cofactor\ matrix\ of\ A]^T$

$$\text{Adj.}A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Ex. 10) Find the inverse of the matrix A where $A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ by using adjoint method.

Solution : refer Ex No. 8 to find adj A.

$$\therefore \quad \text{Adj.}A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$= 3(5 - 4) -2\ (5 - 4) + 6(2 - 2) = 3(1) -2(1) + 0 = 3 - 2 = 1 \neq 0$

$\therefore\ A^{-1}$ exists

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{adj\ A}{1} = adj.A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Ex. 11) Express the following equations in matrix form and solve them by method of reduction.

$x + 3y = 2;\ 3x + 5y = 4$

Solution : The given equations can be written in matrix form as,

$$\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Where $A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Consider AX = B

$\therefore \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Applying $R_2 \to R_2 - 3R_1$

$\begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\begin{bmatrix} x + 3y \\ -4y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

By equality of matrices

$x + 3y = 2$ .... (1)

$-4y = -2$ .... (2)

From (2) $y = \frac{1}{2}$

Substituting $y = \frac{1}{2}$ in (1) we get $x = \frac{1}{2}$

$\therefore x = \frac{1}{2}$ and $y = \frac{1}{2}$ is the required solution.

Ex. 12) Solve the following equations by inversion method.

$x + y = 4;\ 2x - y = 5$

Solution : The given equations can be written in matrix form as,

$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Which is in the form of AX = B

Where $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$, B $= \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$|A| = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = -1 - 2 = -3 \neq 0 \ \therefore A^{-1}$ exists

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}}{-3} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix}$$

We know that $X = A^{-1}B$

$$X = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{9}{3} \\ \frac{3}{3} \end{bmatrix} \quad \therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \therefore\ x = 3,\ y = 1 \text{ is the required solution.}$$

Ex. 13) Express the following equations in matrix form and solve them by method of reduction.

$x + y + z = 6;\ 3x - y + 3z = 6;\ 5x + 5y - 4z = 3$

Solution : The above equations can be written in the form

AX = B

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 3 \\ 5 & 5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix}$$

Using $R_2 - 3R_1$ and $R_3 - 5R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & 0 \\ 0 & 0 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ -27 \end{bmatrix}$$

$$\begin{bmatrix} x + y + z \\ -4y \\ -9z \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ -27 \end{bmatrix}$$

By equality of matrices

$x + y + z = 6$

$-4y = -12 \qquad \Rightarrow y = \frac{-12}{-4} = 3$

$-9z = -27 \qquad \Rightarrow z = \frac{-27}{-9} = 3$

$\therefore\ x + y + z = 6$ becomes

$x + 3 + 3 = 6$

$x = 0$

$\therefore\ x = 0,\ y = 3,\ z = 3$ is the required solution.

Ex. 14) The sum of three numbers is 6. If we multiply third number by 3 and add it to the second number we get 11. By adding first and the third numbers we get a number which is double the second number. Use this information and find a system of linear equations. Find the three numbers using matrices.

Solution : Let the first, Second and third number be x, y, z respectively.

From given data, $x + y + z = 6$

$3z + y = 11 \Rightarrow y + 3z = 11$

$x + z = 2y \Rightarrow x - 2y + z = 0$

Hence the system of linear equations are

$x + y + z = 6$

$y + 3z = 11$

$x - 2y + z = 0$

The above equations can be written in matrix form as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

Applying $R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} x + y + z \\ y + 3z \\ -3y \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -6 \end{bmatrix}$$

By equality of matrices

$x + y + z = 6$

$y + 3z = 11$

$-3y = -6 \quad x = 1$

Solving these we get $x = 1, y = 2, z = 3$

Hence three required numbers are 1, 2, 3.

Exercise 2.1

1) Apply the given elementary transformation on each of the following matrices.

i) $A = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$, $C_1 \leftrightarrow C_2$ ii) $B = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 5 & 4 \end{bmatrix}$, $R_1 \rightarrow R_1 + R_2$

iii) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $2C_2$ iv) $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$, $3R_3$ and then $R_3 + 3R_2$

2) Find the value of k if $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and A (adj A) = KI

3) Find λ if $A = \begin{bmatrix} \lambda & 1 \\ -1 & -\lambda \end{bmatrix}$ and A^{-1} dose not exist.

4) Find which of the following matrices are invertible.

i) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 4 & 3 \\ 1 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

5) If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ Find AB and $(AB)^{-1}$

6) If $A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$ then show that $A^{-1} = \frac{1}{6}(A - 5I)$

7) Find the cofactors of the elements of the following matrices.

i) $\begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$

8) Find the adjoint of the following matrices :

i) $\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

9) Find the inverse of the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ by elementary transformation.

10) Find the inverse of the matrix by adjoint method.

i) $\begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

11) Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ by using a) elementary row transformation b) adjoint method.

12) Solve the following equations by method of reduction.

i) $2x + y = 5$; $3x + 5y = -3$

ii) $x - y + z = 1$; $2x - y = 1$; $3x + 3y - 4z = 2$

13) Solve the following equations by method of inversion.

i) $x + 2y = 2$; $2x + 3y = 3$

ii) $x + y + z = -1$; $y + z = 2$; $x + y - z = 3$

3. Trigonometric Functions

The Principal Solution : The value of variable x which satisfies the trigonometric equation is called principal solution of trigonometric equation where $0 \le x < 2\pi$

Ex., $\sin x = \frac{\sqrt{3}}{2}$ is $x = \frac{\pi}{3}$, and $x = \frac{2\pi}{3}$

1) Find the principal solutions of sin$\theta = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$

Sol. : Given that, $\sin\theta = \frac{1}{\sqrt{2}}$

we know that, $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin\theta = \sin(\pi-\theta)$

$\therefore \sin \frac{\pi}{4} = \sin(\pi - \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

i.e. $\sin \frac{\pi}{4} = \sin \frac{3\pi}{4} = \sin\theta$

here $0 < \frac{\pi}{4} < 2\pi$ and $0 < \frac{3\pi}{4} < 2\pi$

$\therefore \theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ are the principal solutions $\sin\theta = \frac{1}{\sqrt{2}}$

2) Find the principal solutions of cos$\theta = -\frac{1}{2}$

Sol. : Given that, $\cos\theta = -\frac{1}{2}$

We know that, $\cos \frac{\pi}{3} = \frac{1}{2}$ and $-\cos\theta = \cos(\pi - \theta) = \cos(\pi + \theta)$

$\therefore -\cos \frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\pi + \frac{\pi}{3}\right)$

$\therefore -\frac{1}{2} = \cos \frac{2\pi}{3} = \cos \frac{4\pi}{3}$

i.e. $\cos\theta = \cos \frac{2\pi}{3} = \cos \frac{4\pi}{3}$

where $0 < \frac{2\pi}{3} < 2\pi$ and $\frac{4\pi}{3} < 2\pi$

$\therefore \theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$ are the principal solutions $\cos\theta - \frac{1}{2}$

3) Find the principal solutions of tan$\theta = -1$

Sol. : Given that, $\cos\theta = -1$

We know that, $\tan\frac{\pi}{4} = 1$ and $-\tan\theta = \tan(\pi - \theta) = \tan(2\pi - \theta)$

$\therefore -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\left(2\pi - \frac{\pi}{4}\right)$

$\therefore -1 = \tan\frac{3\pi}{4} = \tan\frac{7\pi}{4}$

i.e. $\tan\theta = \tan\frac{3\pi}{4} = \tan\frac{7\pi}{4}$

where $0 < \frac{3\pi}{4} < 2\pi$ and $0 < \frac{7\pi}{4} < 2\pi$

$\therefore \theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$ are the principal solutions $\tan\theta = -1$

The General Solution : The general solution of trigonometric equation the general solution which generalized by using its periodicity.

Ex., General Solution of $\sin x = \frac{\sqrt{3}}{2}$ is $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}$ ......

Theorems : For all $n \in Z$, the general solution of

1) $\sin\theta = 0$ implies$\theta = n\pi$
2) $\cos\theta = 0$ implies$\theta = (2n + 1)$
3) $\tan\theta = 0$ implies$\theta = n\pi$
4) $\sin\theta = \sin a$ implies$\theta = n\pi + (-1)^n a$
5) $\cos\theta = \cos a$ implies$\theta = 2n\pi \pm a$
6) $\tan\theta = \tan a$ implies$\theta = n\pi + a$ and θ and a are even multipal of $\frac{\pi}{2}$
7) $\sin^2\theta = \sin^2 a$ implies$\theta = n\pi \pm a$
8) $\cos^2\theta = \cos^2 a$ implies$\theta = n\pi \pm a$
9) $\tan^2\theta = \tan^2 a$ implies$\theta = n\pi \pm a$
10) $a\cos\theta + b\sin\theta = c$ implies$\theta = 2n\pi \pm a \pm \beta$, $a,b,c \in R$ and $a,b,c \neq 0$

where $\sin a = \frac{b}{\sqrt{a^2 + b^2}}$, $\cos a = \frac{a}{\sqrt{a^2 + b^2}}$, $\cos\beta = \frac{c}{\sqrt{a^2 + b^2}}$

1) Find the general solution of the equation cosec$\theta = -\sqrt{2}$

Sol. : Given that, $= -\sqrt{2}$

$\therefore \sin\theta = -\frac{1}{\sqrt{2}}$ but $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore \sin\theta = -\sin\frac{\pi}{4}$

$\therefore \sin\theta = \sin\left(\pi + \frac{\pi}{4}\right)$ $\because \sin(\pi + \theta) = -\sin\theta$

$\therefore \sin\theta = \sin\left(\frac{5\pi}{4}\right)$

we know that, $\sin\theta = \sin a$ implies$\theta = n\pi + (-1)^n a$

$\therefore \theta = n\pi + (-1)^n\left(\frac{5\pi}{4}\right), n \in Z$

is general solution of trigonometric equation

2) Find the general solution of the equation cos2$\theta = -\frac{1}{2}$

Sol. : Given that, $\cos 2\theta = -\frac{1}{\sqrt{2}}$ but $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore \cos 2\theta = -\cos\frac{\pi}{4}$

$\therefore \cos 2\theta = \cos\left(\pi - \frac{\pi}{4}\right)$ $\because \cos(\pi - \theta) = -\cos\theta$

$\therefore \sin\theta = \sin\left(\frac{3\pi}{4}\right)$

we know that, $\cos\theta = \cos a$ implies$\theta = 2n\pi \pm a$

$\therefore 2\theta = 2n\pi \pm \frac{3\pi}{4}, n \in Z$

$\therefore \theta = n\pi \pm \frac{3\pi}{8}, n \in Z$

is general solution of trigonometric equation

3) Find the general solution of the equation tan3$\theta = -$ 1

Sol. : Given that, $\tan 3\theta = -1$ but $\tan\frac{\pi}{4} = 1$

$$\therefore\ \tan 3\theta = -\tan\frac{\pi}{4}$$

$$\therefore\ \tan 3\theta = \tan\left(\pi - \frac{\pi}{4}\right) \qquad \because\ \tan(\pi - \theta) = -\tan\theta$$

$$\therefore\ \tan 3\theta = \tan\left(\frac{3\pi}{4}\right)$$

we know that, $\tan\theta = \tan a$ implies $\theta = n\pi + a$

$$\therefore\ 3\theta = n\pi + \frac{3\pi}{4},$$

$$\therefore\ \theta = \frac{n\pi}{3} + \frac{n}{4},\ n \in Z$$

is general solution of trigonometric equation

4) Find the general solution of the equation $\cos 5\theta = \sin 3\theta$

Sol. : Given that, $\cos 5\theta = \sin 3\theta$

$$\therefore\ \cos 5\theta = \cos\left(\frac{\pi}{2} - 3\theta\right) \qquad \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

we know that, $\cos\theta = \cos a$ implies $\theta = 2n\pi \pm a$

$$\therefore\ 5\theta = 2n\pi \pm \left(\frac{\pi}{2} - 3\theta\right)$$

$$5\theta = 2n\pi + \left(\frac{\pi}{2} - 3\theta\right) \text{ or } 5\theta = 2n\pi - \left(\frac{\pi}{2} - 3\theta\right)$$

$$5\theta = 2n\pi + \frac{n}{2} - 3\theta) \text{ or } 5\theta = 2n\pi - \left(\frac{\pi}{2} + 3\theta\right)$$

$$\therefore\ 8\theta = 2n\pi + \frac{n}{2} \text{ or } \theta = n\pi - \frac{n}{4}$$

is general solution of trigonometric equation

5) Find the general solution of the equation $4\sin^2\theta = 3$

Sol. : Given that, $4\sin^2\theta = 3$

$$\therefore\ \sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\therefore\ \sin^2\theta = \sin^2\frac{\pi}{3}$$

we know that, $\sin^2\theta = \sin^2 a$ implies $\theta = n\pi \pm a$

$\therefore\ \theta = n\pi \pm \frac{\pi}{3},\ n \in Z$

is general solution of trigonometric equation

6) Find the general solution of the equation $\tan^3\theta - \tan\theta = 0$

Sol. : Given that, $\tan^3\theta - \tan\theta = 0$

$\therefore\ \tan\theta\,(\tan^2\theta - 3) = 0$

$\therefore\ \tan\theta = 0$ or $\tan^2\theta - 3 = 0$

$\therefore\ \tan\theta = 0$ or $\tan^2\theta = \left(\sqrt{3}\right)^2$

$\therefore\ \tan\theta = 0$ or $\tan^2\theta = \tan^2\frac{\pi}{3}$

we know that, $\tan\theta = 0$ implies $\theta = n\pi$ and $\tan^2\theta = \tan^2 a$ implies $\theta = n\pi \pm a$

$\therefore\ \theta = n\pi$ or $\theta = n\pi \pm \frac{\pi}{3},\ n \in Z$

is general solution of trigonometric equation

7) Find the general solution of the equation $\cos\theta - \sin\theta = 1$

Sol. : we know that, if $a \sin x + b \cos x = c$ then divide by $\sqrt{a^2 + b^2}$

Given that, $\cos\theta - \sin\theta = 1$

divide both sides by $\sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\therefore\ \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{\sqrt{2}}$

i.e. $\cos\frac{\pi}{4}\cos\theta - \sin\frac{\pi}{4}\sin\theta = \cos\frac{\pi}{4}$

$\therefore\ \cos\left(\frac{\pi}{4} + \theta\right) = \cos\frac{\pi}{4}$

We know that, $\cos\theta = \cos a$ implies $\theta = 2n\pi \pm a$

$\therefore\ \frac{\pi}{4} + \theta = 2n\pi \pm \frac{\pi}{4}$

$\therefore\ \theta = 2n\pi \pm \frac{\pi}{4} - \frac{\pi}{4}$

$\therefore\ \theta = 2n\pi$ or $\theta = 2n\pi - \frac{\pi}{2}$

is general solution of trigonometric equation

Exercise 3.1

1) Find the principal solutions of following equations.

1) $\cos\theta = \frac{1}{2}$, 2) $\cot\theta = \sqrt{3}$, 3) $\cot\theta = -\sqrt{3}$, 4) $\sin\theta = \frac{1}{2}$

2) Find the general solution of the equation.

1) $\sin\theta = \frac{\sqrt{3}}{2}$, 2) $\cos x = \frac{\sqrt{3}}{2}$, 3) $\cos 4\theta = \cos 2\theta$, 4) $4\cos^2\theta = 3$

5) $\sin\theta = \tan\theta$, 6) $\sec^2 2\theta = 1 - \tan 2\theta$

Polar Co-ordinates :

Cartesian co-ordinate of P be $P(x,y)$ whose polar co-ordinates are $P\ (r,\theta)$

Where $x = r\cos\theta$ and $y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$ i.e. $\theta = \tan^{-1}\frac{y}{x}$ where $0 \le \theta < 2\pi$

1) Find the Cartesian coordinates of pointswhose polar coordinates are$(4,\pi/2)$

Sol. : Given that, polar coordinates are $(r, \theta) = \left(4, \frac{\pi}{2}\right)$

Cartesian co-ordinates are $(x,y) = (r\cos\theta, r\sin\theta)$

$\therefore\ (x,y) = \left(4\cos\frac{\pi}{2}, 4\sin\frac{\pi}{2}\right) = (0,4)$

2) Find the polar coordinates of points whose cartesian coordinates are $\left(-\sqrt{2}, \sqrt{2}\right)$

Sol. : cartesian coordinat es are $(x,y) = \left(-\sqrt{2}, \sqrt{2}\right)$

$\therefore\ r^2 = x^2 + y^2 = (-\sqrt{2})^2 + (\sqrt{2})^2 = 4$ $\therefore\ r = 2$ $\therefore\ r$ always positive

and $\tan\theta = \frac{\sqrt{2}}{-\sqrt{2}} -1$

given point lies second quadrant and $0 \le \theta < 2\pi$

$\therefore\ \tan\theta = -1 \Rightarrow \theta = 135^\circ$ or $\frac{3\pi}{4}$

Polar coordinates are $(r,\theta) = \left(2, \frac{3\pi}{4}\right)$

Theorem : The Sine Rule : If A,B,C are the measures of the angles of a ΔABC and a,b,c are the lengths of sides BC,AC,AB respectively, then

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ where R is the circumradius of ΔABC.

Different forms of Sine rule :

i) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

ii) $a = 2R\ \sin A, b = 2R\ \sin B, c = 2R\ \sin C$

iii) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$

iv) $b\ \sin A = a\ \sin B$, $c\ \sin B = b\ \sin C$ $c \sin A = a\ \sin C$

v) $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}, \dfrac{b}{c} = \dfrac{\sin B}{\sin C}$

Theorem : The Cosine Rule : If A,B,C are the measures of the angles of a ΔABC and a,b,c are the lengths of sides BC, AC, AB respectively, then

1) $a^2 = b^2 + c^2 - 2bc\ \cos A$ OR $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

2) $b^2 = a^2 + c^2 - 2ac\ \cos B$ OR $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

3) $c^2 = a^2 + b^2 - 2ab\ \cos C$ OR $\cos C = \dfrac{a^2 + b^2 - c^2}{2bc}$

Theorem : The Projection Rule : If A,B,C are the measures of the angles of a ΔABC and a,b,c are the lengths of sides BC,AC,AB respectively, then

1) $a = b\ \cos C + c\ \cos B$

2) $b = a\ \cos C + c\ \cos A$

3) $c = a\ \cos B + B\ \cos A$

Application of Sine rule, Cosine rule and Projection rule :

Half Angle Formula :

i) In ΔABC, if $2s = a + b + c$ then

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\ \sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}},\ \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

ii) In ΔABC, if $2s = a + b + c$ then

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},\ \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{bc}},\ \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

iii) In ΔABC, if $2s = a + b + c$ then

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\ \tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}},\ \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Area of Triangle :

The area of a triangle ABC is given by,

$$A(\Delta ABC) = \frac{1}{2}\, ab \sin C = \frac{1}{2}\, ac \sin B = \frac{1}{2}\, bc \sin A$$

Heron's Formula : If a,b,c are the lengths of sides BC,AC,AB of ΔABC and $a + b + c = 2s$, then $A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

Examples on Sine rule, Cosine rule and Projection rule and their Applications :

1) In ΔABC, if $\angle A = 45°$, $\angle B = 60°$ then find the ratio of its sides.

Sol. : Given that, In ΔABC, $\angle A = 45°$, $\angle B = 60°$ and $A+B+C = 180°$ $\therefore$ $C = 75°$

Here, $\sin 60° = \frac{\sqrt{3}}{2}$ $\sin 45° = \frac{1}{\sqrt{2}}$ and

$\sin 75° = \sin(45°+30°) = \sin 45 \cos 30+\cos 45 \sin 30$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

We know that, by sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\therefore \frac{a}{\sin 60°} = \frac{a}{\sin 75°} = \frac{a}{\sin 45°}$$

$$\therefore \frac{a}{\frac{\sqrt{3}}{2}} = \frac{b}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{c}{\frac{1}{\sqrt{2}}} \qquad \therefore a : b : c = \frac{\sqrt{3}}{2} : \frac{\sqrt{3}+1}{2\sqrt{2}} : \frac{1}{\sqrt{2}}$$

$\therefore a : b : c = \sqrt{6} : \sqrt{3}+1 : 2$ (multiply by $2\sqrt{2}$)

2) In ΔABC if $a = 2$, $b = 3$ and $\sin A = \frac{2}{3}$ then find B.

Sol. : By Sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\therefore \frac{2}{\frac{2}{3}} = \frac{3}{\sin B} \quad \sin B = 1 \quad B = 90° = \frac{\pi}{2}$$

3) In ΔABC, if cos then show that ΔABC is an isosceles.

Sol. : We know that, by sine rule, By Sine rule, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$

$\therefore\ a = k \sin A, b = k \sin B$ and $c = k \sin C$

given that $\dfrac{\cos A}{a} = \dfrac{\cos B}{b}$

$$\therefore\ \frac{\cos A}{k \sin A} = \frac{\cos A}{k \sin B}$$

$\therefore\ \cot A = \cot B$

$\therefore\ A = B\ \ \therefore\ \Delta ABC$ is an isosceles

4) Solve the triangle in which $a = 2$, $b = 1$, and $c = \sqrt{3}$

OR

In ΔABC, $a = 2$, $b = 1$, and $c = \sqrt{3}$ then find $\cos A$, $\cos B$, $\cos C$

Sol. : Given that, $a = 2$, $b = 1$, and $c = \sqrt{3}$

We want to find angles A, B, C of ΔABC

We know that, by cosine rule,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1^2 + (\sqrt{3})^2 - 2^2}{2(1)\sqrt{3}} = \frac{1 + 3 - 4}{2\sqrt{3}} = 0$$

$\cos A = 0$ i.e. $A = \dfrac{\pi}{2}$

$$\text{and } \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{2^2 + (\sqrt{3})^2 - 1^2}{2(2)\sqrt{3}} = \frac{4 + 3 - 1}{4\sqrt{3}} = \frac{6}{4\sqrt{3}} = \frac{3}{2\sqrt{3}}$$

$\therefore\ \cos B = \dfrac{\sqrt{3}}{2}$ i.e. $B = \dfrac{\pi}{6}$

$$\text{and } \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{2^2 + 1^2 - (\sqrt{3})^2}{2(2)\sqrt{1}} = \frac{4 + 1 - 3}{4} = \frac{2}{4}$$

5) In ΔABC, Prove that $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 - c^2}{2abc}$

Sol. : $LHS = \dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c}$

By cosine rule,

$$LHS = \frac{\frac{b^2+c^2-a^2}{2bc}}{a} + \frac{\frac{a^2+c^2-b^2}{2ac}}{b} + \frac{\frac{a^2+b^2+c^2}{2ab}}{c}$$

$$= \frac{b^2+c^2-a^2}{2abc} + \frac{a^2+c^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc} = \frac{a^2+b^2+c^2}{2abc} = RHS$$

6) With usual notations, prove that $2\left(a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2}\right) = a + c - b$

Sol. : L.H.S. $= 2\left(a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2}\right)$

$= a\, 2 \sin^2 \frac{C}{2} + c\, 2 \sin^2 \frac{A}{2}$

$= a\,(1 - \cos C) + c\,(1 - \cos A)$

$= a - a \cos C + c - \cos A$

$= a + c - (a \cos \mathrm{C} + c \cos A)$

$= a + c - b =$ R.H.S. (by projection rule)

7) With usual notations, prove that $\frac{c - b \cos A}{b - c \cos A} = \frac{\cos B}{\cos C}$

Sol. : We know that by projection rule,

$a = b \cos C + c \cos B$

$b = a \cos C + c \cos A$

$c = a \cos B + b \cos A$

L.H.S. $= \frac{c - b \cos A}{b - c \cos A}$

$= \frac{a \cos B + b \cos A - b\cos}{a \cos C + c \cos A - c\cos}$

$= \frac{a \cos B}{a \cos C} = \frac{\cos B}{\cos C} = RHS$

8) In ΔABC, if $a = 18$, $b = 24$ and $c = 30$ then

find 1) $\sin \frac{A}{2}$ 2) $\sin \frac{B}{2}$ 3) $\cos \frac{A}{2}$ 4) $\tan \frac{A}{2}$ 5) $A(\Delta ABC)$

Sol. : Given that, $a = 18$, $b = 24$, and $c = 30$

$\therefore\ 2s = a + b + c = 18 + 24 + 30 = 72$ i.e. $s = 36$

1) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}} = \sqrt{\dfrac{(36-24)(36-3)}{\text{bc}}} = \sqrt{\dfrac{(12)\,(6)}{(24)\,(30)}} = \sqrt{\dfrac{(1)\,(1)}{(2)\,(5)}}$

$\therefore \sin\dfrac{A}{2} = \dfrac{1}{\sqrt{10}}$

2) $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{ac}} = \sqrt{\dfrac{(36-1)(36-30)}{(18)\,(30)}} = \sqrt{\dfrac{(18)\,(6)}{(18)\,(30)}} = \sqrt{\dfrac{(1)\,(1)}{(1)\,(5)}}$

$\therefore \sin\dfrac{B}{2} = \dfrac{1}{\sqrt{5}}$

3) $\cos\dfrac{A}{2} = \sqrt{\dfrac{s\,(s-a)}{bc}} = \sqrt{\dfrac{36\,(36-1)}{(24)\,(30)}} = \sqrt{\dfrac{(36)\,(18)}{(24)\,(30)}} = \sqrt{\dfrac{(3)\,(3)}{(2)\,(5)}}$

$\therefore \cos\dfrac{A}{2} = \dfrac{3}{\sqrt{10}}$

4) $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s\,(s-a)}} = \sqrt{\dfrac{(36-2)(36-3)}{36\,(36-18)}} = \sqrt{\dfrac{(12)\,(6)}{(36)\,(18)}} = \sqrt{\dfrac{(1)\,(1)}{(3)\,(3)}}$

$\therefore \tan\dfrac{A}{2} = \dfrac{1}{3}$

5) $A(\Delta ABC) = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$

$= \sqrt{36\,(36-18)\,(36-24)\,(36-30)}$

$= \sqrt{36\,(18)\,(12)\,(6)} = \sqrt{36\,(18)\,(2)\,(3)\,(6)}$

$\therefore A(\Delta ABC) = \sqrt{(36)\,(36)\,(36)} = (6)\,(6)\,(6) = 216$ sq.units

9) In ΔABC, $a\cos^2\dfrac{C}{2} + c\cos^2\dfrac{A}{2} = \dfrac{3b}{2}$ then prove that a,b,c are in A.P.

Sol. : given that, $a\cos^2\dfrac{C}{2} + c\cos^2\dfrac{A}{2} = \dfrac{3b}{2}$

$\therefore a\left(\dfrac{s\,(s-a)}{ab}\right) + c\left(\dfrac{s\,(s-a)}{bc}\right) = \dfrac{3b}{2}$

$\therefore \dfrac{s\,(s-c)}{b} + \dfrac{s\,(s-a)}{b} = \dfrac{3b}{2}$

$\therefore 2s(s-c) + 2s(s-a) = 3b^2$

$\therefore 2s(s-c+s-a) = 3b^2$

$\therefore\ 2s(2s - c - a = 3b^2$

$\therefore\ (a + b + c)\ (a + b + c + c - a) = 3b^2$

$\therefore\ 2s = a + b + c$

$\therefore\ (a + b + c)\ (b) + 3b^2$

$\therefore\ a + b + c = 3b$

$\therefore\ a + c = 2b$ i.e. a,b,c are in A.P.

Exercise 3.2

1) In ΔABC if $A = 30°$, $B = 60°$ then find the ratio of its sides.
2) In ΔABC, if $\sin^2 A + \sin^2 B = \sin^2 C$ then show that ΔABC is a right angled triangle.
3) In ΔABC, prove that $a\ (\sin B - \sin C) + b\ (\sin C - \sin A) + c\ (\sin A - \sin B) = 0$
4) In ΔABC, prove that $(a - b) \sin C + (b - c)\ \sin A + (c - a)\ \sin B = 0$
5) In ΔABC, if $a = 2$, $b = 3$, $c = 4$ then prove that the triangle is obtuse angled.
6) In ΔABC, if $A = 60°$, $b = 3$ and $c = 8$ then find a. Also find the circumradius of the triangle.
7) with usual notations, show that $2(bc \cos A + ac\ cos B + ab \cos C)\ a^2 + b^2 + c^2$
8) With usual notations, prove that $ac \cos B - \text{bc} \cos A = a^2 - b^2$
9) In ΔABC, prove that $(a + b) \cos C + (b + c)\ \cos \text{A} + (c + a)\ \cos B = a + b + c$
10) In ΔABC, show that $\sin \dfrac{A}{2}\ \sin \dfrac{B}{2}\ \sin \dfrac{C}{2} = \dfrac{[A(\Delta ABC)]^2}{abcs}$

Inverse Trigonometric functions :

If $f : X \rightarrow Y$ is such that $f(x) = y$ is one-one and onto function then there exists a unique $f^{-1} : : Y \rightarrow X$ and $f^{-1}(y) = x$ for all $x \in X, y \in$ Y

If $\sin\theta = x$ is trigonometric function, then$\theta = \sin^{-1} x$ is an inverse trigonometric function.

If $\sin\theta = x$ then x is the sine value of angleθ.

If $\theta = \sin^{-1} x$ then $\sin^{-1} x$ is the angle whose value is x.

The following tables gives the domains, ranges and the principal values of all inverse trigonometric functions.

Function	Domain	Range	Principal value branch
$y - \sin^{-1}x$	[−1,1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1}x$	[−1,1]	$[0, \pi]$	$0 \le y \le x$
$y = \tan^{-1}x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \cot^{-1}x$	R	$[0, \pi]$	$0 \le y \le x$
$y = \sec^{-1}x$	R–(–1,1)	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$	$0 \le y \le \pi, y = \frac{\pi}{2}$
$y = \text{cosec}^{-1}x$	R–(–1,1)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}\ y \ne 0$

Properties of Inverse Trigonometric Functions

Property 1

1) $\sin^{-1}\frac{1}{x}\ \text{cosec}^{-1}x$ for $x \ge 1$ or $x \le -1$

2) $\cos^{-1}\frac{1}{x}\ \sec^{-1}x$ for $x \ge 1$ or $x \le -1$

3) $\tan^{-1}\frac{1}{x} = \cot^{-1}x$ for $x > 0$

4) $\text{cosec}^{-1}\frac{1}{x} = \sin^{2\,-1}x$ for $x \ge 1$ or $x \le -1$

5) $\sec^{-1}\frac{1}{x} = \cos^{-1}x$ for $x \ge 1$ or $x \le -1$

6) $\cot^{-1}\frac{1}{x} = \tan^{-1}x$ for $x > 0$

Property 2

1) $\sin^{-1}(-x) = -\sin^{-1}x$ for $-1 \le x \le 1$

2) $\tan^{-1}(-x) = -\tan^{-1}x$ for $x \in R$

3) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$ for $|x| \ge 0$

Property 3

1) $\cos^{-1}(-x) = \pi - \cos^{-1}x$ for $x \in [-1,1]$

2) $\sec^{-1}(-x) = \pi - \sec^{-1}x$ for $|x| \ge 0$

3) $\cot^{-1}(-x) = \pi - \cot^{-1}x$ for $x \in R$

Property 4

1) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ for $x \in [-1,1]$

2) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$ for $x \in R$

3) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \dfrac{\pi}{2}$ for $|x| \geq 0, x \in R$

Property 5

1) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ for $x > 0, y > 0$ and $xy < 1$

2) $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ for $x > 0, y > 0$ and $xy < 1$

3) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

Examples on Inverse trigonometric functions

1) **Find the principal value of $\sin^{-1}\left(\dfrac{-1}{2}\right)$**

Sol. : Let, $\sin^{-1}\left(\dfrac{-1}{2}\right) = x \quad \therefore \sin x = \dfrac{-1}{2}$ but $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$

$\therefore \sin x = \dfrac{-1}{2} = -\sin\dfrac{\pi}{6} = \sin\left(-\dfrac{\pi}{6}\right)$ i.e. $x = -\dfrac{\pi}{6}$

principal value branch of $\sin^{-1}x$ is $\left[\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

here $-\dfrac{\pi}{2} < \dfrac{\pi}{6} < \dfrac{\pi}{2} \quad \therefore x = \dfrac{\pi}{6}$

the principal value of $\sin^{-1}\left(\dfrac{-1}{2}\right)$ is $-\dfrac{\pi}{6}$

2) **Find the value of $\tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2)$**

Sol. : Let, $\tan^{-1}\left(\sqrt{3}\right) = x$ and $\sec^{-1}(-2) = y$

where $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ and $0 < y < \pi, y \neq 0$

$\therefore \tan x = \sqrt{3}$ but $\tan\dfrac{\pi}{3} = \sqrt{3}$ i.e $x = \dfrac{\pi}{3} \quad \therefore \dfrac{\pi}{2} < \dfrac{\pi}{3} < \dfrac{\pi}{2}$

and $\sec y = -2$ but $\sec \frac{\pi}{3} = 2$

$\therefore \sec y = -2 = -\sec \frac{\pi}{3} = \sec\left(\pi - \frac{\pi}{3}\right)$ i.e $y = \frac{2\pi}{3}$ $\therefore 0 < \frac{2\pi}{3} < \pi$

$\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$ and $\sec^{-1}(-2) = \frac{2\pi}{3}$

Now, $\tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = \frac{-\pi}{3}$

3) Find the value of $\sin^{-1}\left(\sin \frac{3\pi}{5}\right)$

Sol. : Let $\sin^{-1}(\sin \frac{3\pi}{5}) = y$

$\sin y = \sin \frac{3\pi}{5}$, where $y \in$

but $y = \frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ The principal value branch of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

we have $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\therefore \sin y = \sin \frac{3\pi}{5} = \sin\left(\pi - \frac{2\pi}{5}\right) = \sin \frac{2\pi}{5}$

$\therefore y = \frac{2\pi}{5}$ and $\frac{2\pi}{5} \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\sin \frac{3\pi}{5}\right) = \frac{2\pi}{5}$

4) Prove that : $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$

Sol. : we know that, $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{(x+y)}{(1-xy)}$

L.H.S. $= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1\frac{11}{23}}\right) = \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}(1) = \frac{\pi}{4} = R.H.S.$$

5) Show that $2\sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{24}{7}\right)$

Sol. : Let, $\sin^{-1}\left(\frac{3}{5}\right) = x$ i.e. $\sin x = \frac{3}{5}$

$$\therefore \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}}$$

$$\therefore \tan x = \sqrt{\frac{\sin x}{\operatorname{Cos} x}} = \frac{3/5}{4/5} = \frac{3}{4} \quad \text{i.e. } x = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\left(\frac{3}{4}\right)$$

Now,

$$2\sin^{-1}\left(\frac{3}{5}\right) = 2\tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{4}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{4}}{1\,\frac{33}{44}}\right) = \tan^{-1}\left(\frac{\frac{6}{4}}{\frac{16-9}{16}}\right) = \tan^{-1}\left(\frac{24}{7}\right)$$

6) Show that $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

Sol. : LHS $= \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1\,\frac{11}{57}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1\,\frac{11}{38}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{7+5}{35}}{\frac{35-1}{35}}\right) + \tan^{-1}\left(\frac{\frac{8+3}{24}}{\frac{24-1}{35}}\right)$$

$$= \tan^{-1}\left(\frac{12}{34}\right) + \tan^{-1}\left(\frac{11}{23}\right) = \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1\,\frac{6\;11}{1723}}\right) + \left(\frac{\frac{138+187}{391}}{\frac{391-66}{391}}\right) = \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}(1) = \frac{\pi}{4} = RHS$$

7) If $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$ **then find the value of** x**.**

Sol. : given that, $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$$\therefore \tan^{-1}(\cos x) + \tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\therefore\ \tan^{-1}\left(\frac{2}{1-\cos^2 x}\right) = \tan^{-1}(2\ \text{cosec}\ x)$$

$$\therefore\ \frac{2\cos x}{1-\cos^2 x} = 2\ \text{cosec}\ x$$

$$\therefore\ \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\therefore\ \frac{\cos x}{\sin x} = 1 \qquad \therefore\ \cot x = 1 \qquad \therefore\ x = \frac{\pi}{4}$$

8) If $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$**, find** x

Sol. : given that,

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\therefore\ \tan^{-1}\left[\frac{\left(\frac{x-1}{x-2}\right)+\left(\frac{x+1}{x+2}\right)}{1-\left(\frac{x-1}{x-2}\right)+\left(\frac{x+1}{x+2}\right)}\right] = \frac{\pi}{4}$$

$$\therefore\ \tan^{-1}\left[\frac{\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)}}{\frac{(x-2)(x+2)-(x-1)(x+1)}{(x-2)(x+2)}}\right] = \frac{\pi}{4}$$

$$\therefore\ \tan^{-1}\left[\frac{(x^2(x-2)+(x^2-x-2)}{(x^2-4)(x^2-1)}\right] = \frac{\pi}{4}$$

$$\therefore\ \tan^{-1}\left[\frac{2x^2-4}{-3}\right] = \frac{\pi}{4}$$

$$\therefore\ \frac{2x^2-4}{-3} = \tan\frac{\pi}{4} = 1$$

$$\therefore\ 2x^2-4 = -3 \qquad \therefore\ 2x^2 = 1$$

$$\therefore\ x^2 = \frac{1}{2} \quad \therefore\ x = \pm\frac{1}{\sqrt{2}}$$

9) **solve the equation $\sin(\sin^{-1}\frac{1}{5} + \cos^{-1}x) = 1$**

Sol. : given that,

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$$

$$\sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}1$$

$$\sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$$

we know that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ $\quad\therefore\ x = \frac{1}{5}$

10) **Prove that $(\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{56}{65}$**

Sol. : Let, $\sin^{-1}\frac{3}{5} = x$ i.e. $\sin x = \frac{3}{5}$

$$\therefore \cos x = \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

and $\cos^{-1}\frac{12}{13} = y$ i.e $.\cos y = \frac{12}{13}$

$$\therefore\ \sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\therefore\ \sin x = \frac{3}{5},\ \cos x = \frac{4}{5},\ \cos y = \frac{12}{13} \text{ and } \sin y = \frac{5}{13}$$

We know that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x+y) = \frac{3}{5}, \frac{12}{13} + \frac{4}{5}, \frac{5}{13} = \frac{36+20}{65}$$

$$\sin(x+y) = \frac{56}{65}$$

$$x+y = \sin^{-1}\frac{56}{65}$$

$$\sin^{-1}\frac{3}{5} = \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{56}{65}$$

11) **Prove that : $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x$ for $\frac{-1}{\sqrt{2}} \le x \le \frac{-1}{\sqrt{2}}$**

Sol. : Let, $x = \sin\theta$ i.e. $\theta = \sin^{-1}x$

$\frac{-1}{\sqrt{2}} \le x \le \frac{-1}{\sqrt{2}}$ $\qquad \therefore\ -1 < 2x\sqrt{1-x^2} < 1$

$\therefore$ L.H.S. $= \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$= \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$= \sin^{-1}(2\sin\theta\cos\theta)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta = 2\sin^{-1}x = R.H.S.$

Exercise 3.3

1) Find the principal value of $\sin^{-1}\frac{1}{\sqrt{2}}$

2) Find the principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$

3) Find the value of $\cos^{-1}\left(\cos\frac{7\pi}{5}\right)$

4) Prove that : $2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4}$

5) Show that $\tan^{-1}\left(\frac{67}{19}\right) + \tan^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{3}{5}\right)$

6) If $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$ then find x.

7) Solve the equation $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$, for $x > 0$

8) Prove that $\tan^{-1}\left[\frac{\cos x + \sin x}{\cos x - \sin x}\right] = \frac{\pi}{4} + x$

9) Show that $\tan^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$ for $\frac{1}{\sqrt{2}} \le x \le 1$

4. Pair of Straight Lines

Introduction :

We know that equation $ax + by + c = 0$, where $a, b, c \in R$, (a and b not zero simultaneously), represents a line in XY plane. We are familiar with different forms of equations of line. Now let's study two lines simultaneously. For this we need the concept of the combined equation of two lines.

4.1 Combined equation of a pair of lines :

Definition : An equation which represents two lines is called the combined equation of those two lines.

Let $u \equiv a_1x + b_1y + c_1$ and $v \equiv a_2x + b_2y + c_2$. Equations $u = 0$ and $v = 0$ represent lines. We know that equation $u + kv = 0$, $k \in R$ represents a family of lines. Let us interpret the equation $uv = 0$.

Homogeneous Equation : An equation in which the degree of every term is same, is called a homogeneous equation.

For Example :

$x^2 + 3xy = 0$, $7xy - 2y^2 = 0$, $5x^2 + 3xy - 2y^2 = 0$ are homogeneous equations.

Theorem 1 : The equation $uv = 0$ represents the combined equation of lines $u = 0$ and $v = 0$

Theorem 2 : The combined equation of a pair of lines passing through the origin is a homogeneous equation of degree two in x and y

Theorem 3 : The acute angle θ between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$

The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin if $h^2 - ab \geq 0$.

Remarks :

1) If $h^2 - ab > 0$ then lines represented by (1) are distinct.
2) If $h^2 - ab = 0$ then lines represented by (1) are coincident.
3) If $h^2 - ab < 0$ then equation (1) does not represent a pair of lines.
4) If $b = 0$ then one of the lines is the Y- axis, whose slope is not defined and the slope of the other line is $-\frac{a}{2h}$ (provided that $h \neq 0$).

5) If $h^2 - ab \geq 0$ and $b \neq 0$ then slopes of the lines are $m_1 = \dfrac{-h-\sqrt{h^2 - ab}}{b}$ and

$m_2 = \dfrac{-h+\sqrt{h^2 - ab}}{b}$

Their sum is $m_1 + m_2 = -\dfrac{2h}{b}$ and product is $m_1 m_2 = \dfrac{a}{b}$

The quadratic equation in m whose roots are m_1 and m_2 is given by

$m^2 - (m_1 + m_2)m + m_1 m_2 = 0$

$\therefore\ m^2 - \left(-\dfrac{2h}{b}\right) m + \dfrac{a}{b} = 0 \qquad \therefore\ bm^2 + 2hm + a = 0 \ldots (2)$

Equation (2) is called the auxiliary equation of equation (1). Roots of equation (2) are slopes of lines represented by equation (1).

General Second Degree Equation in x and y :

Equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, where at least one of a, b, h is not zero, is called a general second degree equation in x and y.

Remarks : The necessary conditions for a general second degree equation.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of lines are :

i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

ii) $h^2 - ab \geq 0$

Remarks : If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines then

1) These lines are parallel to the lines represented by $ax^2 + 2hxy + by^2 = 0$.

2) The acute angle between them is given by $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a + b}\right|$

3) Condition for lines to be perpendicular to each other is $a + b = 0$.

4) Condition for lines to be parallel to each other is $h^2 - ab = 0$.

5) Condition for lines to intersect each other is $h^2 - ab > 0$ and the co-ordinates of their point of intersection are $\left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$.

6) The expression $abc + 2fgh - af^2 - bg^2 - ch^2$ is the expansion of the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

7) The joint equation of the bisectors of the angles between lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $hx^2 - (a - b)xy - hy^2 = 0$.

Here the coefficient of x^2 + coefficient of $y^2 = 0$. Hence bisectors are perpendicular to each other.

Ex. 1) Find the combined equation of lines $x - 2y = 0$ and $x + y = 0$.

Solution : The combined equation of lines $u = 0$ and $v = 0$ is $uv = 0$.

$\therefore$ The combined equation of lines $x - 2y = 0$ and $x + y = 0$ is

$(x - 2y)(x + y) = 0$

$\therefore x^2 - xy - 2y^2 = 0$

Ex. 2) Find separate equations of lines represented by $x^2 - y^2 + x + y = 0$.

Solution : We factorize equation $x^2 - y^2 + x + y = 0$ as

$(x + y)(x - y) + (x + y) = 0$

$\therefore (x + y)(x - y + 1) = 0$

Required separate equations are $x + y = 0$ and $x - y + 1 = 0$.

Ex. 3) Find the sum of slopes of lines represented by $x^2 + 4xy - 7y^2 = 0$

Solution : Comparing equation $x^2 + 4xy - 7y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$,

we get $a = 1$, $h = 2$ and $b = -7$.

If m_1 and m_2 are slopes of lines represented by this equation then

$m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1 m_2 = \dfrac{a}{b}$

$\therefore m_1 + m_2 = -\dfrac{4}{-7} = \dfrac{4}{7}$

Their sum is $\dfrac{4}{7}$

Ex. 4) Find the angle between lines represented by $3x^2 - 4xy - 3y^2 = 0$ are.

Solution : Comparing given equation with $ax^2 + 2hxy + by^2 = 0$,

we get $a = 3$, $h = -2$ and $b = -3$.

As $a + b = 3 + (-3) = 0$, lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other therefore angle between these two lines is $90°$

Ex. 5) Find the combined equation of lines $x + y - 2 = 0$ and $2x - y + 2 = 0$.

Solution : The combined equation of lines $u = 0$ and $v = 0$ is $uv = 0$.

$\therefore$ The combined equation of lines $x + y - 2 = 0$ and $2x - y + 2 = 0$ is

$(x + y - 2)(2x - y + 2) = 0$

$\therefore x\ (2x - y + 2) + y\ (2x - y + 2) - 2\ (2x - y + 2) = 0$

$\therefore 2x^2 - xy + 2x + 2xy - y^2 + 2y - 4x + 2y - 4 = 0$

$\therefore 2x^2 + xy - y^2 - 2x + 4y - 4 = 0$

Ex. 6) Show that lines represented by equation $x^2 - 2xy - 3y^2 = 0$ are distinct.

Solution : Comparing equation $x^2 - 2xy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 1$, $h = -1$, and $b = -3$.

$h^2 - ab = (-1)^2 - (1)(-3) = 1 + 3 = 4 > 0$

As $h^2 - ab > 0$, lines represented by equation $x^2 - 2xy - 3y^2 = 0$ are distinct.

Ex. 7) Find the angle between pair of lines represented by $x^2 - 4xy + y^2 = 0$

Solution : Comparing equation $x^2 - 4xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$,

we get $a = 1$, $h = -2$ and $b = 1$.

Let θ be the acute angle between them.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{2\sqrt{4 - 1}}{2}\right| = \sqrt{3}$$

$$\therefore \theta = 60° = \frac{\pi}{3}$$

Ex. 8) Find the value of k if $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$

Solution : Slope of the line $2x + y = 0$ is -2.

As $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$, -2 is a root of the auxiliary equation $2m^2 + km + 3 = 0$.

$\therefore 2\ (-2)^2 + k\ (-2) + 3 = 0$

$\therefore 8 - 2k + 3 = 0$

$\therefore\ -2k + 11 = 0$

$\therefore\ 2k = 11$

$$\therefore\ k = \frac{11}{2}$$

Ex. 9. Find the combined equation of the pair of lines passing through the origin and perpendicular to the lines represented by $3x^2 + 2xy - y^2 = 0$.

Solution : Let m_1 and m_2 are slopes of lines represented by $3x^2 + 2xy - y^2 = 0$.

$$\therefore m_1 + m_2 = -\frac{2h}{b} = -\frac{2}{-1} = 2 \text{ and } m_1m_2 = \frac{a}{b} = \frac{3}{-1} = -3$$

Now required lines are perpendicular to given lines.

$\therefore$ Their slopes are $-\dfrac{1}{m_1}$ and $\dfrac{1}{m_2}$.

And required lines pass through the origin.

$\therefore$ Their equations are $y = -\dfrac{1}{m_1}x$ and $y = -\dfrac{1}{m_2}x$

$\therefore m_1y = -x$ and $m_2y = -x$

$\therefore x + m_1y = 0$ and $x + m_2y = 0$

Their combined equation is $(x + m_1y)(x + m_2y) = 0$

$\therefore x^2 + (m_1 + m_2)\, xy + m_1m_2y^2 = 0$

$\therefore x^2 + (2)\, xy + (-3)\, y^2 = 0$

$\therefore x^2 + 2xy - 3y^2 = 0$

Ex. 10) Find the combined equation of lines passing through the origin and each of which making angle 60° with the X-axis.

Solution : Let m be the slope of one of the required lines.

The slope of the X-axis is 0. As required lines make angle 60° with the X-axis,

$$\tan 60° = \left|\frac{m - 0}{1 + (m)\,(0)}\right|$$

$\therefore \sqrt{3} = |m| \quad \therefore m^2 = 3$

$\therefore m^2 + 0m - 3 = 0$ is the auxiliary equation.

$\therefore$ The required combined equation is $-3x^2 + 0xy + y^2 = 0$

$\therefore 3x^2 - y^2 = 0$

Ex. 11) Find the value of k if the equation $2x^2 + 4xy - 2y^2 + 4x + 8y + k = 0$ represents a pair of lines.

Solution : Comparing given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get, $a = 2,\ b = -2,\ c = k$

$f = 4,\ g = 2,\ h = 2$

As given equation represents a pair of lines, it must satisfy the necessary condition.

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\therefore (2)(-2)(k) + 2(4)(2)(2) - 2(4)^2 - (-2)\,(2)^2 - (k)\,(2)^2 = 0$

$\therefore -4k + 32 - 32 + 8 - 4k = 0$

$\therefore 8k = 8 \quad \therefore k = 1$

Exercise 4.1

1) Find the combined equation of the following pairs of lines : $2x + y = 0$ and $3x - y = 0$

2) Find joint equation of coordinate axes.

3) Find joint equation of lines $x + y = 0$ and $x - y = 0$

4) Find separate equation of lines represented by $x^2 - 4y^2 = 0$

5) Find the value of k if sum of slope of lines given by $3x^2 + kxy - y^2 = 0$ is zero.

6) Find the value of k if the lines represented by $kx^2 + 4xy - 4y^2 = 0$ are perpendicular.

7) Show that equation $x^2 + 2xy + 2y^2 + 2x + 2y + 1 = 0$ does not represent a pair of lines.

8) Find the joint equation of lines

(i) Passing through the origin and having slopes 2 and 3.

(ii) Passing through the origin and having inclinations $60°$ and $120°$

(iii)Passing though (1, 2) and parallel to the co-ordinate axes.

9) Find separate equation of lines represented by $3x^2 -10xy - 8y^2 = 0$

10) Find k if the sum of slopes of the lines given by $x^2 + kxy - 3y^2 = 0$ is equal to their product.

11) Find the measure of the acute angle between the lines represented by $3x^2 - 4\sqrt{3}\,xy + 3y^2 = 0$

12) Find the combined equation of the following pairs of lines passing through $(-1, 2)$, one is parallel to $x + 3y - 1 = 0$ and the other is perpendicular to $2x - 3y -1 = 0$.

13) Find the separate equations of the lines represented by

$$(x - 2)^2 - 3\,(x - 2)(y + 1) + 2(y +1)^2 = 0$$

14) Find the value of k if the equation : $3x^2 + 10xy + 3y^2 + 16y + k = 0$ represent a pair of lines

15) Find the combined equation of a pair of lines passing through the origin and perpendicular to the lines $5x^2 - 8xy + 3y^2 = 0$

16) Find k if slopes of lines represented by $3x^2 + kxy - y^2 = 0$ differ by 4.

17) Find the condition that the line $4x + 5y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$.

18) If slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is four times the other then show that $16h^2 = 25ab$.

19) Find the combined equation of lines passing through the origin and each of which making angle $60°$ with the X-axis.

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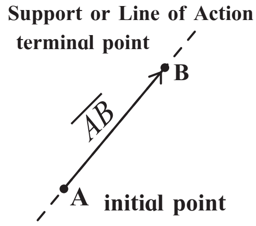
5. Vectors

Introduction :

1) **Vectors :** The physical quantity which is completely described by magnitude as well as direction is known as vector quantity.



Ex. Force, velocity, acceleration etc.

- A is initial point
- B is terminal point
- Direction is from A to B
- Distance between A and B is magnitude, Denoted by $|\overline{AB}|$ or AB

2) **Unit vector :** The vector whose magnitude is 1 is known as unit vector.

- unit vector of $\overline{a}$ is $\hat{a}$ $\qquad \therefore \hat{a} = \dfrac{\overline{a}}{|\overline{a}|}$
- unit vector along X–axis is $\hat{i}$
- unit vector along Y–axis is $\hat{j}$
- unit vector along Z–axis is $\hat{k}$

3) **Position vector :**

- position vector of point A is $\overline{OA}$, denoted by $\overline{a}$.
- If A (x, y, z) then $\overline{a} = x\hat{i} + y\hat{j} + z\hat{k}$
- If A (x_1, y_1, z_1) and B (x_2, y_2, z_2) then

$\therefore\ \overline{AB} = \overline{b} - \overline{a} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$\therefore\ \overline{AB}\ (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$l(AB) = |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

4) **Distance of p(x, y, z) from :**

i) X–axis is $\sqrt{y^2 + z^2}$ $\qquad$ ii) Y–axis is $\sqrt{x^2 + z^2}$

iii) Z–axis is $\sqrt{x^2 + y^2}$ $\qquad$ iv) XY plane is $|z|$

v) YZ plane is $|x|$ $\qquad$ vi) ZX plane is $|y|$

vii) Origin is $\sqrt{x^2 + y^2 + z^2}$

Theorem 1 : Two nonzero vectors $\overline{a}$ and $\overline{b}$ are colinear if and only if there exist scalars m and n, at least one of them is nonzero such that $m\overline{a} + n\overline{b} = \overline{0}$.

Theorem 2 : Let $\overline{a}$ and $\overline{b}$ be non–colinear vectors. A vector $\overline{r}$ is Coplanar with $\overline{a}$ and $\overline{b}$ if and only if there exist unique scalars t_1, t_2 such that $\overline{r} = t_1\ \overline{a} + t_2\overline{b}$.

Theorem 3 : Three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar, if and only if there exists a nonzero linear combination $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ with $(x, y, z) \neq (0, 0, 0)$

Theorem 4 : If $\bar{a}$, $\bar{b}$, $\bar{c}$ are three non–coplanar vectors then any vector $\bar{r}$ in the space can be uniquely expressed as a linear combination of $\bar{a}$, $\bar{b}$, $\bar{c}$.

Ex. 1) Determine the value of c that satisfy $|c\bar{u}| = 3$, if $\bar{u} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Solution : $\therefore$ Given $\bar{u} = \hat{i} + 2\hat{j} + 3\hat{k}$

Multiplying by c

$\therefore c\bar{u} = c\hat{i} + 2c\hat{j} + 3c\hat{k}$

Taking magnitude

$\therefore |c\bar{u}| = \sqrt{c^2 + 4c^2 + 9c^2}$

$\therefore \quad 3 = \sqrt{14c^2}$

$\therefore \quad c^2 = \dfrac{9}{14}$

$c = \pm \dfrac{3}{\sqrt{14}}$

Ex. 2) Find the unit vector along the direction of $8\hat{i} + 3\hat{j} - \hat{k}$.

Solution : Given vector, $\bar{a} = 8\hat{i} + 3\hat{j} - \hat{k}$

$\therefore |\bar{a}| = \sqrt{64 + 9 + 1}$

$|\hat{a}| = \sqrt{74}$

unit vector along the direction of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{8\hat{i} + 3\hat{j} - \hat{k}}{\sqrt{74}}$$

Ex. 3) If the vectors $2\hat{i} - 9\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear then find the value of q.

Solution : Given vectors $\bar{a} = 2\hat{i} - 9\hat{j} + 3\hat{k}$ and

$\bar{b} = 4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear.

$\therefore \dfrac{2}{4} = \dfrac{-9}{-5} = \dfrac{3}{6}$

$\therefore \dfrac{1}{2} = \dfrac{q}{4}$

$\therefore q = \dfrac{5}{2}$

Ex. 4) Show that point $A(3, 2, -4)$, $B(9, 8, -10)$, $C(-2, -3, 1)$ are collinear.

Solution : Given point $A(3, 2,-4)$, $B(9, 8,-10)$, $C(-2,-3, 1)$ are collinear.

$\therefore\ \overline{AB} = \overline{b} - \overline{a} = (9\hat{i} + 8\hat{j} - 10\hat{k}) - (3\hat{i} + 2\hat{j} - 4\hat{k})$

$\therefore\ \overline{AB} = 6\hat{i} + 6\hat{j} - 6\hat{k}$ ...(I)

and $\overline{AC} = \overline{c} - \overline{a} = (-2\hat{i} - 3\hat{j} + \hat{k}) - (3\hat{i} + 2\hat{j} - 4\hat{k})$

$\overline{AC} = -5\hat{i} -5\hat{j} + 5\hat{k}$

$\overline{AC} = -5(\hat{i} + \hat{j} - \hat{k})$ Multiplying and dividing by 6

$\overline{AC} = -\dfrac{5}{6}(6\hat{i} + 6\hat{j} - 6\hat{k})$

$\therefore\ \overline{AC} = \dfrac{-5}{6}\overline{AB}$ ...(by I)

$\overline{AC}$ is scalar multiple of $\overline{AB}$

$\overline{AC} \parallel \overline{AB}$ since point A is common

$\therefore$ points A, B, C are collinear.

Ex. 5) Find the distance of $P(4, -2, 6)$ from (i) XY plane (ii) Z–axis

Solution : Given point $P(4, -2, 6)$

(i) Distance of P from XY plane $= |z|$

$= |6| = 6$ unit

(ii) Distance of P from Z axis $= \sqrt{(x^2 + y^2)}$

$= \sqrt{16 + 4)}$

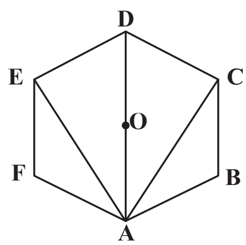
$= \sqrt{20}$

$= 2\sqrt{5}$ unit

Ex. 6) If $ABCDEF$ is a regular hexagon show that

$\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 3\overline{AD} = 6\ \overline{AO}$ where O is centre of Hexagon.

Solution : $ABCDEF$ is regular hexagon



$\overline{ED} = \overline{AB}$, $\overline{CD} = \overline{AF}$ ...(*i*)

O is centre of hexagon,

$\overline{AD} = 2\overline{AO}$ ...(*ii*)

In ΔACD, $\overline{AC} + \overline{CD} = \overline{AD}$ ...(*iii*) (Triangle Law)

In ΔAED, $\overline{AE} + \overline{ED} = \overline{AD}$ ...(*iv*) (Triangle law)

Adding (iii) and (iv)

$\overline{AC} + \overline{CD} + \overline{AE} + \overline{ED} = 2\overline{AD}$

$\therefore \overline{AC} + \overline{AF} + \overline{AE} + \overline{AB} = 2\overline{AD}$ (By i)

Adding $\overline{AD}$ in both sides

$\therefore \overline{AC} + \overline{AF} + \overline{AE} + \overline{AB} + \overline{AD} = 3\overline{AD}$

$\therefore \overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 3\overline{AD} = 3(2\overline{AO})$ (By ii)

$\therefore \overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 3\overline{AD} = 6\overline{AO}$

Hence proved,

Theorem 5 : (*section formula internal division*)

Let $A(\overline{a})$ and $B(\overline{b})$ be any two points in the space and $R(\overline{r})$ be a point on the line segment AB dividing it

internally in the ratio $m : n$ then, $\overline{r} = \dfrac{m\overline{b} + n\overline{a}}{m + n}$

Theorem 6 : (*section formula external division*)

Let $A(\overline{a})$ and $B(\overline{b})$ be any two points in the space and $R(\overline{r})$ be the third point on line AB dividing the segment

AB externally in the ratio $m : n$ then $\overline{r} = \dfrac{m\overline{b} - n\overline{a}}{m - n}$.

Remarks :

i) Midpoint formula : If $M(\overline{m})$ is midpoint of line segment Joining $A(\overline{a})$ and $B(\overline{b})$ then $\overline{m} = \dfrac{(\overline{a} + \overline{b})}{2}$

ii) Centroid of ΔABC : $\overline{g} = \dfrac{\overline{a} + \overline{b} + \overline{c}}{3}$

iii) Centroid of tetrahedron $ABCD$ is $\overline{g} = \dfrac{\overline{a} + \overline{b} + \overline{c} + \overline{d}}{4}$

iv) In center Formula of ΔABC : $\overline{I} = \dfrac{(BC)\ \overline{a} + (AC)\ \overline{b} + (AB)\ \overline{c}}{BC + AC + AB}$

iv) To find the ratio of division : It is convenient to consider $k : 1$ (Instead of $m : n$)

v) Collinearity of 3 – points : Three distinct points A, B and C with position vectors $\overline{a}, \overline{b}, \overline{c}$ respectively are collinear if and only if there exist three non–zero scalars x, y and z such that $x\overline{a} + y\overline{b} + z\overline{c} = \overline{0}$ and $x + y + z = 0$

Ex. 7) Find the position vector of point *R* which divides the line segment joining $P(\overline{P}) = 2\hat{i} - \hat{j} + 3\hat{k}$ and $Q(\hat{q}) = -5\hat{i} + 2\hat{j} - 5\hat{k}$ internally in the ratio 1:3

Solution : $R(\overline{r})$ divides the line segment joining points $P(\overline{p})$ and $Q(\overline{q})$ internally in the rate 1 : 3

By section formula internal division

$$\overline{r} = \frac{1\overline{q} + 3\overline{p}}{1 + 3} \quad \ldots\text{(using } \overline{r} = \frac{m\overline{b} + n\overline{a}}{m + n} \text{ formula)}$$

$$\overline{r} = \frac{1(-5\hat{i} + 2\hat{j} - 5\hat{k}) + 3(2\hat{i} - \hat{j} + 3\hat{k})}{1 + 3}$$

$$\overline{r} = \frac{5\hat{i} + 2\hat{j} - 5\hat{k} + 6\hat{i} - 3\hat{j} + 3\hat{k})}{4}$$

$$\overline{r} = \frac{\hat{i} - \hat{j} + 4\hat{k}}{4}$$

$$\overline{r} = \frac{1}{4}\hat{i} - \frac{1}{4}\hat{j} + \hat{k}$$

Ex. 8) If $\overline{a}$, $\overline{b}$, $\overline{c}$ are position vectors of the points *A*, *B*, *C* respectively and $5\overline{a} - 3\overline{b} - 2\overline{c} = \overline{0}$ find the ratio in which *C* divides seg *BA*.

Solution : Given $5\overline{a} - 3\overline{b} - 2\overline{c} = \overline{0}$

$$5\overline{a} - 3\overline{b} = 2\overline{c}$$

$$2\overline{c} = 5\overline{a} - 3\overline{b}$$

$$\overline{c} = \frac{5\overline{a} - 3\overline{b}}{2}$$

$$\overline{c} = \frac{5\overline{a} - 3\overline{b}}{5 - 3}$$

This shows that, point *C* divides segment *BA* externally in the ratio 5 : 3.

Ex. 9) If *G*(a, 2, –1) is centroid of the triangle with vetefices *P*(1, 2, 3), *Q*(3, b, –4) and *R*(5, 1, c) then find the values of a, b, c.

Solution : As $G(\overline{g})$ is centroid of –PQR

$$\overline{g} = \frac{\overline{p} + \overline{q} + \overline{r}}{3}$$

$$\therefore\ a\hat{i} + 2\hat{j} - \hat{k} = \frac{(\hat{i} + 2\hat{j} + 3\hat{k}) + (3\hat{i} + b\hat{j} - 4\hat{k}) + (5\hat{i} + \hat{j} - c\hat{k})}{3}$$

$$\therefore\ 3a\hat{i} + 6\hat{j} - 3\hat{k} = 9\hat{i} + (3 + b)\hat{j} + (c - 1)\hat{k}$$

By equating like terms, we set

$\therefore 3a = 9,\ 3 + b = 6,\ c - 1 = -3$

$\therefore\ a = 3,\ b = 3,\ c = -2$

Ex. 10) Find the centroid of tetrahedron whose vertices are *K*(5, −7, 0), *L*(1, 5, 3), M(4, −6, 3), N(6, −4, 2)

Solution : Position vectors of vertices of tetrahedron $\overline{k} = 5\hat{i} - 7\hat{j} + 0\hat{k}$,

$\overline{l} = \hat{i} + 5\hat{j} + 3\hat{k},\ \overline{m} = 4\hat{i} - 6\hat{j} + 3\hat{k},\ \overline{n} = 6\hat{i} - 4\hat{j} + 2\hat{k}$

By Centroid formula :

$$\overline{g} = \frac{\overline{k} + \overline{l} + \overline{m} + \overline{n}}{4}$$

$$\overline{g} = \frac{(5\hat{i} - 7\hat{j} + 0\hat{k}) + (\hat{i} + 5\hat{j} + 3\hat{k}) + (4\hat{i} - 6\hat{j} + 3\hat{k}) + (6\hat{i} - 4\hat{j} + 2\hat{k})}{4}$$

$$\overline{g} = \frac{16\hat{i} - 12\hat{j} + 8\hat{k}}{4}$$

$$\overline{g} = 4\hat{i} - 3\hat{j} + 2\hat{k}$$

$\therefore$ Centroid of tetrahedron G ≡ (4, −3, 2)

Ex. 11) Find the position vector of mid–point *M* joining the points *L*(7, −6, 12) and *N*(5, 4, −2).

Solution : *M* is midpoint of segment *LN*. By midpoint formula

$$\overline{m} = \frac{\overline{l} + \overline{n}}{2}$$

$$\overline{m} = \frac{(7\hat{i} - 6\hat{j} + 12\hat{k}) + (5\hat{i} + 4\hat{j} - 2\hat{k})}{4}$$

$$\overline{m} = \frac{12\hat{i} - 2\hat{j} + 10\hat{k}}{2}$$

$$\overline{m} = 6\hat{i} - \hat{j} + 5\hat{k}$$

Scalar product of two vectors (Dot product) :

Let $\overline{a}$, $\overline{b}$ be any two vectors and θ be angle between them then the dot product of $\overline{a}$ and $\overline{b}$ is $\overline{a} \cdot \overline{b} = |\overline{a}|\,|\overline{b}| \cos\theta$

i) If $\overline{a} \perp \overline{b}$ then $\overline{a} \cdot \overline{b} = 0$

ii) $\overline{a} \cdot \overline{a} = |\overline{a}|^2$

iii) $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$, $\quad \hat{i} \cdot \hat{j} = 0$, $\hat{j} \cdot \hat{k} = 0$, $\hat{k} \cdot \hat{i} = 0$

iv) If $\bar{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\bar{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ then $\bar{a} \cdot \bar{b} = x_1x_2 + y_1y_2 + z_1 z_2$

v) Angle between $\bar{a}$ and $\bar{b}$ is $\theta = \cos^{-1}\left(\dfrac{\bar{a} \cdot \bar{b}}{|\bar{a}|\,|\bar{b}|}\right)$

vi) Scalar projection of $\bar{a}$ on $\bar{b} = \dfrac{\bar{a} \cdot \bar{b}}{|\bar{b}|}$ and Vector projection of $\bar{a}$ on $\bar{b} = \left(\dfrac{\bar{a} \cdot \bar{b}}{|\bar{b}|^2}\right)\bar{b}$

vii) If α, β, γ be angles made by vector with X,Y,Z axes respectively then α, β, γ are known as direction angles and $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ are known as **direction cosines** (d.c.s.).

viii)Three real numbers a, b, c are known as direction ratios if $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c}$.

ix) If a, b, c are direction ratios of line then

$$l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Ex. 12) If $|\bar{a}| = 3$, $|\bar{b}| = \sqrt{6}$ and angle between $\bar{a}$ and $\bar{b}$ is 45°, then find $\bar{a} \cdot \bar{b}$

Solution : Given $|\bar{a}| = 3$, $|\bar{b}| = \sqrt{6}$, $\theta = 45°$.

$$\bar{a} \cdot \bar{b} = |\bar{a}|\,|\bar{b}| \cos\theta$$

$$= (3)(\sqrt{6})\cos\theta\ 45°$$

$$= 3\sqrt{6} \times \frac{1}{\sqrt{2}}$$

$$= 3\sqrt{3}\ \sqrt{2}\ \frac{1}{\sqrt{2}}$$

$$\bar{a} \cdot \bar{b} = 3\sqrt{3}$$

Ex. 13) If $\bar{a} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $|\bar{b}| = 3\hat{i} - 4\hat{j} + 5\hat{k}$ find

i) $\bar{a} \cdot \bar{b}$ $\qquad$ ii) angle between $\bar{a}$ and $\bar{b}$

Solution : Given vectors :

$\bar{a} = 3\hat{i} + 4\hat{j} - 5\hat{k}$, $\bar{b} = 3\hat{i} - 4\hat{j} + 5\hat{k}$,

i) $\bar{a} \cdot \bar{b} = (3\hat{i} + 4\hat{j} - 5\hat{k}) \cdot (3\hat{i} - 4\hat{j} + 5\hat{k})$

$= (3)(3) + 4\,(-4) + (-5)\,(5) = -32$

$\bar{a} \cdot \bar{b} = -32$

ii) Let θ be angle between $\overline{a}$ and $\overline{b}$

$$\cos\theta = \frac{\overline{a} \cdot \overline{b}}{|\overline{a}|\,|\overline{b}|} = \frac{-32}{\sqrt{9+16+25}\ \sqrt{9+16+25}}$$

$$\cos\theta = \frac{-32}{50} = \frac{-16}{25}$$

$$= \cos^{-1}\left(\frac{-1}{25}\right)$$

Ex. 14) Find the value of a if $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} + a\hat{j} - 3\hat{k}$ are perpendicular.

Solution : Given vectors :

$\overline{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$, $\overline{b} = \hat{i} + a\hat{j} - 3\hat{k}$ are perpendicular

$\overline{a} \cdot \overline{b} = 0$

$\therefore (3\hat{i} + 2\hat{j} + 9\hat{k}).(\hat{i} + a\hat{j} - 3\hat{k}) = 0$

$\therefore 3 + 2a + 27 = 0$

$\therefore \quad 2a = -30$

$a = -15$

Ex. 15) Find the scalar projection and vector projection of $3\hat{i} + 4\hat{j} - 6\hat{k}$ on $2\hat{i} - 2\hat{j} + \hat{k}$

Solution : Given Vectors

$\overline{a} = 3\hat{i} + 4\hat{j} - 6\hat{k}$, $\overline{b} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\overline{a} \cdot \overline{b} = (3\hat{i} + 4\hat{j} - 6\hat{k}).(2\hat{i} - 2\hat{j} + \hat{k}) = 0$

$= (3)(3) + 4\,(-2) + (-6)\,(1)$

$\therefore \quad = 6 + 8 + 6$

$\therefore \overline{a} \cdot \overline{b} = -8$

and $|\overline{b}| = \sqrt{4+4+1} = \sqrt{9}$

$|\overline{b}| = 3$

Scalar projection of $\overline{a}$ on $\overline{b} = \dfrac{\overline{a} \cdot \overline{b}}{|\overline{b}|} = \dfrac{-8}{3}$

vector projection of $\overline{a}$ along $\overline{b} = \dfrac{\overline{a} \cdot \overline{b}}{|\overline{b}|^2}\overline{b}$

$$= \frac{-8}{9}(2\hat{i} - 2\hat{j} + \hat{k})$$

Ex. 16) A line makes angles of measure 45° and 60° with positive direction of Y and Z–axes respectively. And the angle made by the line with the positive direction of the X–axis.

Solution : Let $\alpha : \beta, \gamma$ be angles made by line with positive direction of X, y and z axes respectively

Given $\beta = 45°, \gamma = 60°$

$\cos\beta = \cos 45°$ and $\cos\gamma = \cos 60°$

$\therefore \cos\beta = \dfrac{1}{\sqrt{2}}$ and $\cos\gamma = \dfrac{1}{2}$

we have, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$$

$$\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$$

$$\cos^2\gamma = 1 - \frac{3}{4}$$

$$\cos^2\gamma = \frac{1}{4}$$

$$\cos\gamma = \pm\frac{1}{2}$$

$$\gamma = 60° \text{ or } 120°$$

Ex. 17) A line has direction ratios 4, –12, 18 then find the direction cosines.

Solution : Direction ratios of line

(a, b, c) = (4, –12, 18)

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{16 + 144 + 324} = \sqrt{484}$$

$$\sqrt{a^2 + b^2 + c^2} = 22$$

Direction cosines of line

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \pm\left(\frac{4}{22}\right),\ m = \pm\left(\frac{-12}{22}\right),\ n = \pm\left(\frac{18}{22}\right)$$

$$l = \left(\frac{2}{11}\right),\ m = \left(\frac{-6}{11}\right),\ n = \left(\frac{9}{11}\right)$$

OR $l = \dfrac{-2}{11},\ m = \dfrac{6}{11},\ n = \dfrac{-9}{11}$

Vector product of two vectors :

Vector product of $\overline{a}$ and $\overline{b}$ denoted by $a \times \overline{b}$:

It is defined as $\overline{a} \times \overline{b} = |\overline{a}|\,|\overline{b}| \sin\theta\, \hat{n}$

Note : i) $\hat{n}$ is unit vector a long $\overline{a} \times \overline{b}$

ii) $\overline{a} \times \overline{b}$ is perpendicular to both $\overline{a}$ and $\overline{b}$.

Formulae :

i) $\overline{a} \times \overline{a} = \overline{0}$

ii) $\overline{b} \times \overline{a} = -(\overline{a} \times \overline{b})$

iii) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \overline{0}$

iv) $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$

v) If $\overline{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\overline{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$,

$$\overline{a} \times \overline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

vi) Unit vector perpendicular to both $\overline{a}$ and $\overline{b}$ is $\hat{n} = \dfrac{\overline{a} \times \overline{b}}{(|\overline{a}|\,|\overline{b}|)}$

vii) $\sin\theta = \dfrac{|\overline{a} \times \overline{b}|}{(|\overline{a}|\,|\overline{b}|)}$

viii) If $\overline{a}$, $\overline{b}$ be any two vectors along adjacent sides of parallelogram. then Area of parallelogram $= |\overline{a} \times \overline{b}|$

ix) If $\overline{a}$, $\overline{b}$ be along diagonals of parallelogram then Area of parallelogram $= \dfrac{1}{2}|\overline{a} \times \overline{b}|$

x) $A(\Delta ABC) = \dfrac{1}{2}|\overline{AB} \times \overline{AC}|$

xi) $|\overline{u} \times \overline{v}|^2 + |\overline{u} \cdot \overline{v}|^2 = |\overline{u}|^2\, |\overline{v}|^2$

Ex. 18) If $\overline{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\overline{b} = \hat{i} - 4\hat{j} + 2\hat{k}$ then find $(\overline{a} + \overline{b}) \times (\overline{a} - \overline{b})$

Solution : Given vectors

$\overline{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\overline{b} = \hat{i} - 4\hat{j} + 2\hat{k}$

$(\overline{a} + \overline{b}) = 3\hat{i} - \hat{j} + \hat{k}$, $(\overline{a} - \overline{b}) = \hat{i} + 7\hat{j} - 3\hat{k}$

Now,

$$(\bar{a}+\bar{b})\times(\bar{a}-\bar{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & -4 & 2 \end{vmatrix}$$

$$= \hat{i}(6-4) - \hat{j}(4+1) + \hat{k}(-8-3)$$

$$(\bar{a}+\bar{b})\times(\bar{a}-\bar{b}) = 2\hat{i} - 5\hat{j} - 11\hat{k}$$

Ex. 19) Find unit vector perpendicular to vectors $\hat{j}+2\hat{k}$ and $\hat{i}+\hat{j}$

Solution : Given vectors, $\bar{a} = \hat{j}+2\hat{k}$, $\bar{b} = \hat{i}+\hat{j}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \hat{i}(0-2) - \hat{j}(0-2) + \hat{k}\,(0-1)$$

$$\bar{a}\times\bar{b} = -2\hat{i}+2\hat{j}-\hat{k}$$

$$|\bar{a}\times\bar{b}| = \sqrt{(-2)^2+(2)^2+(-1)^2} = \sqrt{4+4+1} = \sqrt{9}$$

$$|\bar{a}\times\bar{b}| = 3$$

Unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ is

$$\hat{n} = \frac{\bar{a}\times\bar{b}}{(|\bar{a}|\,|\bar{b}|)}$$

$$\hat{n} = \frac{-2\hat{i}+2\hat{j}-\hat{k}}{3}$$

$$\hat{n} = -\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

Ex. 20) If $|\bar{u}| = 2$, $|\bar{v}| = 5$, $|\bar{u}\times\bar{v}| = 8$ find $\bar{u}\cdot\bar{v}$

Solution : Given $|\bar{u}| = 2$, $|\bar{v}| = 5$, $|\bar{u}\times\bar{v}| = 8$

Now, we have,

$$|\bar{u}\times\bar{v}|^2 + (\bar{u}\cdot\bar{v})^2 = |\bar{u}|^2\cdot|\bar{v}|^2$$

$$\therefore (8)^2 + \;= (\bar{u}\cdot\bar{v})^2 = (2)^2 - (5)^2$$

$$\therefore\; 64 + (\bar{u}\cdot\bar{v})^2 = 100$$

$$\therefore\; (\bar{u}\cdot\bar{v})^2 = 36$$

$$\therefore\; \bar{u}\cdot\bar{v} = \pm 6$$

Ex. 21) Find the area of parallelogram whose adjacent sides are along vectors $\bar{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{b} = \hat{i} - 3\hat{j} + 3\hat{k}$

Solution : Vectors along adjacent sides of parallelogram are

$\bar{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{b} = \hat{i} - 3\hat{j} + 3\hat{k}$

Now, $|\bar{a} \times \bar{b}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -3 & 3 \end{vmatrix}$

$= \hat{i}(6 + 3) - \hat{j}(-6 - 1) + \hat{k}(-6 + 2)$

$\bar{a} \times \bar{b} = 9\hat{i} + 7\hat{j} - 4\hat{k}$

$|\bar{a} \times \bar{b}| = \sqrt{(9)^2 + (7)^2 + (-4)^2}$

$= \sqrt{81 + 49 + 16}$

$= \sqrt{146}$

$|\bar{a} \times \bar{b}| = 3$

Area of parallelogram $= |\bar{a} \times \bar{b}|$

$= \sqrt{146}$ sq.units

Scalar Triple Product (Box product) :

The scalar product of $\bar{a}$ and $(\bar{b} \times \bar{c})$ is known as scalar triple product

Denoted by $\bar{a} \, . \, (\bar{b} \times \bar{c})$ or $[\bar{a} \, (\bar{b} \; \bar{c})]$

Note : 1) If $\bar{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{c}$, $\bar{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, $\bar{c} = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$ then

$$[\bar{a} \; \bar{b} \; \bar{c})] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

2) If at least two vectors in scalar triple product are collinear then value of scalar triple product is zero, i.e. $[\bar{a} \; \bar{a} \; \bar{b}] = 0$.

3) If $\bar{a}, \bar{b}, \bar{c}$ are coplanar vectors then, $[\bar{a} \; \bar{b} \; \bar{c}] = 0$

Theorem 7 : If $\bar{a}, \bar{b}, \bar{c}$ be three vectors along coterminus edges of parallelopiped then volume of parallelopiped, $V = [\; \bar{a} \; \bar{b} \; \bar{c} \;]$

Theorem 8 : If $\bar{a}, \bar{b}, \bar{c}$ be three vectors along coterminus edges of tetrahedron then volume of tetrahedron, $v = \frac{1}{6} [\bar{a} \; \bar{b} \; \bar{c}]$

Vector triple product : $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c}) \, \bar{b} - (\bar{a} \cdot \bar{b}) \, \bar{c}$

Ex. 23) **If the vectors $-3\hat{i} + 4\hat{j} - 2\hat{k}$, $\hat{i} + 2\hat{k}$ and $\hat{i} - \hat{j}$ are coplanar then find the value of p.**

Solution : Vectors $\overline{a} = -3\hat{i} + 4\hat{j} - 2\hat{k}$, $\overline{b} = \hat{i} + 2\hat{k}$, $\overline{c} = \hat{i} - p\hat{j}$ are coplanar.

$\therefore \overline{a} \cdot (\overline{b} \times \overline{c}) = 0$

$$\begin{vmatrix} -3 & 4 & -2 \\ 1 & 0 & 2 \\ 1 & -p & 0 \end{vmatrix} = 0$$

$\therefore -3(0 + 2p) - 4(0 - 2) + (-2)(-p - 0) = 0$

$\therefore -6p + 8 + 2p = 0$

$\therefore \quad 4p = 8$

$\therefore \quad p = 2$

Ex. 24) **If $\overline{c} = 3\overline{a} - 2\overline{b}$ then show that $[\overline{a}\ \overline{b}\ \overline{c}] = 0$.**

Solution : $[\overline{a}\ \overline{b}\ \overline{c}] = \overline{a} \cdot (\overline{b} \times \overline{c})$

$= \overline{a} \cdot (\overline{b} \times (3\overline{a} - 2\overline{b}))(\because \overline{c} = 3\overline{a} - 2\overline{b})$

$= \overline{a} \cdot (3(\overline{b} \times \overline{a}) - 2(\overline{b} \times \overline{b}))$

$= \overline{a} \cdot (3(\overline{b} \times \overline{a}) - 2 \cdot \overline{b})$

$= 3\overline{a} \cdot (\overline{b} \times \overline{a})$

$= 3(0)$...(Two vectors are equal)

$[\overline{a}\ \overline{b}\ \overline{c}] = 0$

Exercise 5.1

1) Show that vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

2) Show that the three points $A(1, -2, 3)$, $B(2, 3, -4)$ and $C(0, -7, 10)$ are collinear.

3) Find a vector of magnitude 7 units along the direction of $\overline{a} = \hat{i} - 2\hat{j} + 2\hat{k}$.

4) Find the distance of $P(3, 4, 6)$ from (1) XY plane(2) Z–axis.

5) If vectors $\hat{i} + 2\hat{j} + \lambda\hat{k}$ and $3\hat{i} + 6\hat{j} + 9\hat{k}$ are collinear then, find the value of λ.

6) If three points $A(3, 2, P)$, $B(9, 8, -10)$, $C(-2, -3, 1)$ are collinear find (1) the ratio in which the point C divides the line segment AB (ii) the values of p and q.

7) If $A(5, 1, p)$, $B(1, q, p)$ and $C(1,-2, 3)$ are vertices of ΔABC and $G\left(r, -\frac{4}{3}, \frac{1}{3}\right)$ is its centroid then find the values of p, q and r.

8) Find the position vector of point which divides the join of $P(\overline{p}) = 2\hat{i} - \hat{j} + 3\hat{k}$ and $Q(\overline{q}) = -5\hat{i} + 2\hat{j} - 5\hat{k}$ externally in the ratio 3 : 2.

9) If $\overline{a} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\overline{b} = 3\hat{i} - 4\hat{j} - 5\hat{k}$ find $\overline{a} \cdot \overline{b}$.

10) Find the value of λ if $\overline{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, and $\overline{b} = \lambda\hat{i} - 2\hat{j} + 4\hat{k}$ are perpendicular.

11) Find the measure of acute angle between vectors $2\hat{i} + \hat{j} + \hat{k}$ and $\hat{i} - \hat{j} + 2\hat{k}$.

12) Find the scalar projection of $\hat{i} + 2\hat{j} + 3\hat{k}$ on $2\hat{i} - 2\hat{j} + \hat{k}$

13) Find the vector projection of $\hat{i} + 2\hat{j} + 3\hat{k}$ along $2\hat{i} - 2\hat{j} + \hat{k}$

14) Find the direction cosines of the vector $2\hat{i} + 2\hat{j} - \hat{k}$

15) If a line makes angles 90°, 135°, 45° with X, Y and Z axes respectively then find its direction cosines.

16) If $\overline{a} = \hat{i} + \hat{j} - \hat{k}$, $\overline{b} = 2\hat{i} + 4\hat{j} + 6\hat{k}$ then find $|\overline{a} \times \overline{b}|$

17) If $|\overline{a}| = 5$, $|\overline{b}| = 13$ and $|\overline{a} \times \overline{b}| = 25$ find $\overline{a} \cdot \overline{b}$

18) Find the area of parallelogram whose diagonals are determined by the vectors $\overline{a} = 3\hat{i} - \hat{j} - 2\hat{k}$, $\overline{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$

19) Find the direction ratios of a vector perpendicular to two lines whose direction ratios are 1, 3, 2 and −1, 1, 2

20) Find the volume of parallelopiped determined by the vectors $\overline{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\overline{b} = -\hat{i} + \hat{j} + 2\hat{k}$, $\overline{c} = 2\hat{i} + \hat{j} + 4\hat{k}$.

21) If vectors $\hat{i} + 5\hat{j} - 2\hat{k}$, $3\hat{i} - \hat{j}$, $5\hat{i} + m\hat{j} - 4\hat{k}$ are coplanar then find the value of m.

22) If D, E, F are midpoint of sides BC, CA, AB of ΔABC then, prove that $\overline{AD} + \overline{BE} + \overline{CF} = \overline{O}$

23) If $\overline{a} = \hat{i} + \hat{j}$, $\overline{b} = \hat{j} + \hat{k}$, $\overline{c} = \hat{i} + \hat{j} + \hat{k}$ find $[\overline{a}\ \overline{b}\ \overline{c}]$.

24) If α, β, γ be direction angles of vector then show that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

25) Using vector method, find the co–ordinates of the triangle whose vertices are $A(0, 3, 0)$, $B(0, 0, 4)$ $C(0, 3, 4)$.

26) Find $\overline{a} \cdot (\overline{b} \times \overline{c})$ if $\overline{a} = 3\hat{i} - \hat{j} + 4\hat{k}$, $\overline{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\overline{c} = -5\hat{i} + 2\hat{j} + 3\hat{k}$.

27) Find the volume of a tetrahedron whose vertices are $A(-1, 2, 3)$, $B(3, -2, 1)$, $C(2, 1, 3)$ $D(-1, -2, 4)$.

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6. Line and Plane

I) Line :

1) Passing through a point and parallel to vector.

- The vector equation of the line passing through A($\overline{a}$) and parallel to vector $\overline{b}$ is $\overline{r} = \overline{a} + \lambda\overline{b}$
- equation $\overline{r} = \overline{a} + \lambda\overline{b}$ is also called the parametric form of vector equation of line.
- $(\overline{r} - \overline{a}) \times \overline{b} = \overline{O}$ This equation is called the non-parametric form of vector equation of line.
- The Cartesian equations of the line passing through $A(x_1, y_1, z_1)$ and having direction ratios a,b,c are $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$
- The equations $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ are called the **symmetric form** of Cartesian equations of line.
- The equations $x = x_1 + \lambda a$, $y = y_1 + \lambda b$, $z = z_1 + \lambda c$ are called **parametric form** of the Cartesian equations of line.
- The co-ordinates of variable point P on the line are $(x_1 + \lambda a, x_1 + \lambda b, z_1 + \lambda c)$
- Corresponding to each real value of λ there is one point on the line and conversely corresponding to each point on the line there is unique real value of λ.

2) Passing through two points.

- The equation of the line passing through $A(\overline{a})$ and $B(\overline{b})$ is $\overline{r} = \overline{a} + \lambda(\overline{b} - \overline{a})$.
- The Cartesian equations of the line passing through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

3) Distance of a point from a line.

- The distance of point P($\overline{\alpha}$) from the line $\overline{r} = \overline{a} + \lambda\overline{b}$ is $\sqrt{|\overline{\alpha} - \overline{a}|^2 - \left[\dfrac{(\overline{\alpha} - \overline{a}).\overline{b}}{|\overline{b}|}\right]}$

4) Skew lines

A pair of lines in space which neither intersect each other nor are parallel to each other are called skew lines. Skew lines are non-coplanar

5) **Distance between skew lines**

- The distance between lines $\overline{r} = \overline{a}_1 + \lambda \overline{b}_1$ and $\overline{r} = \overline{a}_2 + \lambda_2 \overline{b}_2$ is
$$\left| \frac{(\overline{a}_2 - \overline{a}_1).(\overline{b}_1 \times \overline{b}_2)}{|\overline{b}_1 \times \overline{b}_2|} \right|$$

6) **Distance between parallel lines.**

- The distance between parallel lines $\overline{r} = \overline{a}_1 + \lambda \overline{b}_1$ and $\overline{r} = \overline{a}_2 + \lambda_2 \overline{b}_2$ is $| \overline{a}_2 - \overline{a}_1 \times \overline{b} |$

II) Plane :

1) **Equation of a plane passing through a point and perpendicular to a vector.**

- The equation of the plane passing through the point $A(\overline{a})$ and perpendicular to vectorn is $\overline{r} \cdot \overline{n} = \overline{a} \cdot \overline{n}$
- Equation $\overline{r} \cdot \overline{n} = \overline{a} \cdot \overline{n}$ is called the **vector equation of plane in scalar product form.**
- If $\overline{a} \cdot \overline{n} = d$ then equation $\overline{r} \cdot \overline{n} = \overline{a} \cdot \overline{n}$ takes the form $\overline{r} \cdot \overline{n} = d$

2) **Cartesian form :**

- The equation of the plane passing through the point $A(x_1, y_1, z_1)$ and direction ratios of whose normal are a,b,c is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
- The equation of the plane passing through a point and parallel to two vectors.
- The vector equation of the plane passing through the point $A(\overline{a})$ and parallel to non-zero and non-parallel vectors $\overline{b}$ and $\overline{c}$ is $(\overline{r} - \overline{a}) \cdot (\overline{b} \times \overline{c}) = 0$
- $\overline{r} = \overline{a} + \lambda \overline{b} + \mu \overline{c}$ This equation is called the vector equation of plane in parametric form.

3) **Passing through three non-collinear points :**

- The vector equation of the plane passing through non-collinear points $A(\overline{a})$, $B(\overline{b})$ and $C(\overline{c})$ is $(\overline{r} - \overline{a}). [(\overline{b} - \overline{a}) \times (\overline{b} - \overline{a})] = 0$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y - y_1 & z - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

The normal form of equation of plane.

- The equation of the plane at distance p unit from the origin and to which uni vector $\hat{n}$ is normal is $\overline{r} \cdot \hat{n} = p$.

- If l, m, n are direction cosines of the normal to a plane then $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$.
- If N is the foot of the perpendicular drawn from the origin to the plane and $ON=p$ then the co-ordinates of N are (pl,pm,pn).
- The equation of the plane is $lx+my+nz=p$. This is the **normal form of the Cartesian equation** of the plane.

Equation of plane passing through the intersection of two planes :

If planes $(\overline{r} \cdot \overline{n}_1 - \overline{d}_1) = 0$ and $\overline{r} \cdot \overline{n}_2 = 0$ intersect each other, then for every real value of λ, equation $\overline{r} \cdot (\overline{n}_1 - \lambda \overline{n}_2) - (\overline{d}_1 - \lambda \overline{d}_2) = 0$ represents a plane passing through the line of their intersection. If planes $a_1x+b_1y+c_1z+d_1 =0$ and $a_2x+b_2y+c_2z+d_2 =0$ intersect each other, then for every real value of λ, equation $(a_1x+b_1y+c_1z+d_1) + \lambda (a_2x+b_2y+c_2z+d_2) = 0$ represents a plane passing through the line of their intersection.

Angle between two planes.

$$\sin\theta = \left| \frac{\overline{b} \cdot \overline{n}}{|\overline{b}|\,|\overline{n}|} \right|$$

Co-planarity of two lines.

Thus lines $\overline{r} = \overline{a}_1 + \lambda_1 \overline{b}_1$ and $\overline{r} = \overline{a}_2 + \lambda_2 \overline{b}_2$ are coplanar if and only if

$(\overline{a}_2 - \overline{a}_1) \cdot (\overline{b}_1 \times \overline{b}_2) = 0$

The plane determined by them passes through $A(\overline{a}_1)$ and $\overline{b}_1 \times \overline{b}_2$ is normal to the plane.

$\therefore$ Its equation is $(\overline{r} - \overline{a}_1) \cdot (\overline{b}_1 \times \overline{b}_2) = 0$

Lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ are coplanar if and only if

$$\begin{vmatrix} x - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

The equation of the plane determined by them is $\begin{vmatrix} x - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

9) **Distance of a point from a plane.**

- The distance of the point A($\overline{a}$) from the plane $\overline{r}\cdot\hat{n}=p$ is given by $|p-|\overline{a}\,.\,\hat{n}||$.

Ex. 1) Find the vector equation of the line passing through the point having position vector $4\hat{i}-\hat{j}+2\hat{k}$ and parallel to vector $-2\hat{i}+\hat{j}+\hat{k}$.

Solution :

The equation of the line passing through $A(\overline{a})$ and parallel to vector $\overline{b}$ is $\overline{r}=\overline{a}+\lambda$ $\overline{b}$

The equation of the line passing through $4\hat{i}-\hat{j}+2\hat{k}$ and parallel to vector $-2\hat{i}+\hat{j}+$ $\hat{k}$ is $\overline{r}=(4\hat{i}-\hat{j}+2\hat{k})+\lambda(-2\hat{i}+\hat{j}+\hat{k})$

Ex. 2) Find the vector equation of the line passing through the point having position vector $2\hat{i}+\hat{j}-3\hat{k}$ and perpendicular to vectors $\hat{i}+\hat{j}-\hat{k}$.

Solution :

Let $\overline{a}=2\hat{i}+\hat{j}-3\hat{k}$, $\overline{b}=\hat{i}+\hat{j}-\hat{k}$ and $\overline{c}=\hat{i}+2\hat{j}-k$

We know that $\overline{b}\times\overline{c}$ is perpendicular to both $\overline{b}$ and $\overline{c}$.

$\therefore\ \overline{b}\times\overline{c}$ is parallel to the required line.

$$\overline{b}\times\overline{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 1 & 2 & -1\end{vmatrix}-3\hat{i}+2\hat{j}+\hat{k}$$

Thus required line passes through $\overline{a}=2\hat{i}+\hat{j}-3\hat{k}$ and parallel to $3\hat{i}+\hat{j}-\hat{k}$.

$\therefore$ Its equation is $\overline{r}=(2\hat{i}+\hat{j}\ -3k)+\lambda(3\hat{i}+2\hat{j}+k)$.

Ex. 3) Find the vector equation of the line passing through $2\hat{i}+\hat{j}-\hat{k}$ and parallel to the line joining points $-\hat{i}+\hat{j}\ -4k$ and $\hat{i}+2\hat{j}-2k$

Solution :

Let A, B, C be points with position vectors $\overline{a}=2\hat{i}+\hat{j}-\hat{k},\ \overline{b}=\hat{i}+2\hat{j}+4k$ and $\overline{c}=\hat{i}+2\hat{j}-2\hat{k}$, respectively.

$\overline{BC}=\overline{c}-\overline{b}=(\hat{i}+2\hat{j}+2\hat{k})-(-\hat{i}+\hat{j}-4\hat{k})=2\hat{i}+\hat{j}-2\hat{k}$

The required line passes through $2\hat{i}+\hat{j}-\hat{k}$ and Parallel to $2\hat{i}+\hat{j}-2\hat{k}$

Its equation is $\overline{r}=(2\hat{i}+\hat{j}+\hat{k})+\lambda(2\hat{i}+\hat{j}+2\hat{k})$

Ex. 4) Find the vector equation of the line passing through $A(1,2,3)$ and $B(2,3,4)$.

Solution :

Let position vectors of points A and B be $\overline{a}$ and $\overline{b}$.

$\therefore\ \overline{a} = \hat{i}+2\hat{j}+3\hat{k}$ and $\overline{b} = 2\hat{i}+3\hat{j}+4\hat{k}$

$\therefore \overline{b}-\overline{a} = (2\hat{i}+3\hat{j}+4\hat{k}) - (\hat{i}+2\hat{j}+3\hat{k}) = \hat{i}+\hat{j}+\hat{k}$

The equation of the line passing through $A(\overline{a})$ and $B(\overline{b})$ is $\overline{r} = \overline{a}+\lambda(\overline{b}-\overline{a})$.

The equation of the required line is $\overline{r} = (\hat{i}+2\hat{j}+3\hat{k}) + \lambda(\hat{i}+\hat{j}+\hat{k})$

Ex. 5) **Find the Cartesian equations of the line passing through A(1,2,3) and having direction ratios 2, 3, 7.**

Solution :

The Cartesian equations of the line passing through $A(x_1, y_1, z_1)$ and having $A(x_2, y_2, z_2)$ direction ratios

a,b,c is $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$

Here $(x_1, y_1, z_1) \equiv (1, 2, 3)$ and direction ratios are 2, 3, 7.

Required equation is $\dfrac{x-1}{2} = \dfrac{y-1}{3} = \dfrac{z-3}{7}$

Ex. 6) **Find the Cartesian equations of the line passing through $A(1, 2, 3)$ and $B(2, 3, 4)$.**

Solution :

Find the Cartesian equations of the line passing through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$

Here $(x_1, y_1, z_1) \equiv (1, 2, 3,)$ and $(x_2, y_2, z_2) \equiv (2, 3, 4)$

Required Cartesian equations are $\dfrac{x-1}{2-1} = \dfrac{y-2}{3-2} = \dfrac{z-3}{4-3}$

$\therefore\ \dfrac{x-1}{1} = \dfrac{y-2}{1} = \dfrac{z-3}{1}$

$\therefore\ x-1 = y-2 = z-3$

Exercise 6.1

1) Find the vector equation of the line passing through the point having position vector $-2\hat{i}+\hat{j}+\hat{k}$ and parallel to vector $4\hat{i}-\hat{j}+2\hat{k}$.

2) Find the vector equation of the line passing through points having position vectors $3\hat{i}+4\hat{j}-7\hat{k}$ and $6\hat{i}-\hat{j}+k$.

3) Find the vector equation of line passing through the point having position vector $5\hat{i}+4\hat{j}+3\hat{k}$ and having direction ratios $-3,4,2$.

4) Find the vector equation of the line passing through the point having position vector $\hat{i}+2\hat{j}+3\hat{k}$ and perpendicular to vectors $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}-\hat{j}+\hat{k}$.

5) Find the vector equation of the line passing through the point having position vector $-\hat{i}-\hat{j}+2\hat{k}$ and parallel to the line r = $(\hat{i}+2\hat{j}+3\hat{k})+(3\hat{i}+2\hat{j}+\hat{k})$.

6) Find the Cartesian equations of the line passing through $A(2,2,1)$ and $B(1,3,0)$.

7) Show that the line $\dfrac{x-2}{1}=\dfrac{y-4}{2}=\dfrac{z+4}{-2}$ passes through the origin.

Ex. 1) Find the length of the perpendicular drawn from the point $P(3,2,1)$ to the line.
$\overline{r}=(7\hat{i}+7\hat{j}+6\hat{k})+\lambda(-2\hat{i}+2\hat{j}+3\hat{k})$

Solution :

The length of the perpendicular is same as the distance of P from the given line. The distance of point $P(\alpha)$ from the line $\overline{r}=\overline{a}+\lambda\overline{b}$ is

$$\sqrt{|\overline{\alpha}-\overline{a}|^2-\left[\frac{(\overline{\alpha}-\overline{a}).\overline{b}}{|\overline{b}|}\right]}$$

Here $\overline{\alpha}=3\hat{i}+2\hat{j}+\hat{k}$, $\overline{a}=7\hat{i}+7\hat{j}+6\hat{k}$, $\overline{b}=2\hat{i}+2\hat{j}+3\hat{k}$

$$\overline{\alpha}-\overline{a}=(3\hat{i}+2\hat{j}+\hat{k})-(7\hat{i}+7\hat{j}+6\hat{k})=-4\hat{i}-5\hat{j}-5\hat{k}$$

$$\therefore\quad |\overline{\alpha}-\overline{a}|=\sqrt{(-4)^2+(5)^2+(-5)^2}=\sqrt{16+25+25}=\sqrt{66}$$

$$(\overline{\alpha}-\overline{a}).\overline{b}=(-4\hat{i}-5\hat{j}-5\hat{k}).(-2\hat{i}+2\hat{j}+3\hat{k})=8-10-15=-17$$

$$|\overline{b}|=\sqrt{(-2)^2+(2)^2+(3)^2}=\sqrt{17}$$

The require length $=\sqrt{|\overline{\alpha}-\overline{a}|^2-\left[\dfrac{(\overline{\alpha}-\overline{a}).\overline{b}}{|\overline{b}|}\right]}=\left[\dfrac{-17}{\sqrt{17}}\right]^2$

$$=\sqrt{66-17}=\sqrt{49}=7 \text{ unit}$$

Ex. 2) Find the distance of the point $P(0, 2, 3)$ from the line. $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$

Solution :

Let M be the foot of the perpendicular drawn from the point P(0,2,3) to the given line.

M lies on the line. Let co-ordinates of M be $(5\lambda-3, 2\lambda+1, 3\lambda-4)$. The direction ratios of PM are $(5\lambda-3)-0$, $(2\lambda+1)-2$, $(3-4)-3$

i.e. $5\lambda-3$, $2\lambda-1$, $3\lambda-7$

The direction ratios of given line are 5, 2, 3 and PM is perpendicular to the given line.

$\therefore\ 5(5\lambda-3), +2(2\lambda-1), 3(3\lambda-7) = 0$

$\therefore\ \lambda = 1$

$\therefore$ The co-ordinates of M are $(2, 3, -1)$.

The distance of P from the line is $PM = \sqrt{(2-0)^2+(3-2)^2+(-1-3)^2} = \sqrt{21}$ unit

Ex. 3) Find the shortest distance between lines $\overline{r} = (2\hat{i}-\hat{j})+(2\hat{i}+\hat{j}-3\hat{k})$ and $\overline{r} = (\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+\hat{j}-5\hat{k})$.

Solution :

The shortest distance between lines $\overline{r} = \overline{a}_1+\lambda_1\overline{b}_1$ and $\overline{r} = \overline{a}_2+\lambda_2\overline{b}_2$ is

$$\left|\frac{(\overline{a}_2-\overline{a}_1).(\overline{b}_1\times\overline{b}_2)}{|\overline{b}_1\times\overline{b}_2|}\right|$$

Here $\overline{a}_1 = 2\hat{i}-\hat{j}, \overline{a}_2 = \hat{i}-\hat{j}+2\hat{k}, \overline{b}_2 = 2\hat{i}-\hat{j}+5\hat{k}$

$\therefore\ \overline{a}_2-\overline{a}_1 = (\hat{i}-\hat{j}+2\hat{k})-(2\hat{i}-\hat{j}) = -\hat{i}+2\hat{k}$

and $\overline{b}_1\times\overline{b}_2 = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -3\\ 2 & 2 & -5\end{vmatrix} = 2\hat{i}+4\hat{j}$

$\therefore\ |\overline{b}_1\times\overline{b}_2| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

$(\overline{a}_2-\overline{a}_1).(\overline{b}_1\times\overline{b}_2) = (-\hat{i}+2\hat{k}).(-2\hat{i}+4\hat{j})$

$\therefore$ The require shortest distance $= \left|\frac{(\overline{a}_2-\overline{a}_1).(\overline{b}_1\times\overline{b}_2)}{|\overline{b}_1\times\overline{b}_2|}\right| = \left|\frac{2}{2\sqrt{5}}\right| = \left|\frac{1}{\sqrt{5}}\right|$ unit

Ex. 4) Show that lines $\overline{r} = (\hat{i}+\hat{j}+\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$ and $\overline{r} = (4\hat{i}+3\hat{j}+2\hat{k})+\mu(\hat{i}-2\hat{j}+2\hat{k})$ intersect each other.

Solution :

Lines $\overline{r} = \overline{a}_1+\lambda_1\overline{b}_1$ and $\overline{r} = \overline{a}_2+\lambda_2\overline{b}_2$ intersect each other if and only if

$(\overline{a}_2-\overline{a}_1).(\overline{b}_1\times\overline{b}_2) = 0$

Here $\overline{a}_1 = \hat{i}+\hat{j}-\hat{k}$, $\overline{a}_2 = 4\hat{i}-3\hat{j}+2\hat{k}$, $\overline{b}_2 = \hat{i}-2\hat{j}+2\hat{k}$

$\overline{a}_2-\overline{a}_1 = 3\hat{i}-4\hat{j}+3\hat{k}$

$$(\overline{a}_2-\overline{a}_1).(\overline{b}_1\times\overline{b}_2) = \begin{vmatrix} 3 & -4 & 3 \\ 2 & -2 & 1 \\ 2 & -2 & 2 \end{vmatrix} = 3(-2)+4(3)+3(-2) = -6+12-6 = 0$$

Given lines intersect each other.

Ex. 5) Find the distance between parallel lines $\overline{r} = (2\hat{i}-\hat{j}+\hat{k})+\lambda(2\hat{i}+\hat{j}-2\hat{k})$ and $\overline{r} = (\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+\hat{j}-2\hat{k})$

Solution :

The distance between parallel lines $\overline{r} = \overline{a}_1+\lambda\overline{b}$ and $\overline{r} = \overline{a}_2+\lambda\overline{b}$ is given by

$$d = |(\overline{a}_2-\overline{a}_1)\times\hat{b}|$$

Here $\overline{a}_1 = 2\hat{i}-\hat{j}+\hat{k}$, $\overline{a}_2 = \hat{i}-\hat{j}+2\hat{k}$, $\overline{b} = 2\hat{i}+\hat{j}-2\hat{k}$

$\therefore$ $\overline{a}_2-\overline{a}_1 = (\hat{i}-\hat{j}+2\hat{k}) - (2\hat{i}-\hat{j}+\hat{k}) = -\hat{i}+\hat{k}$ and $\hat{b} = \dfrac{2\hat{i}-\hat{j}+2\hat{k}}{3}$

$$\therefore\ (\overline{a}_2-\overline{a}_1)\times\hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \end{vmatrix} = \frac{1}{3}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \frac{1}{3}(-\hat{i}-\hat{k})$$

$$d = |(\overline{a}_2-\overline{a}_1)\times\hat{b}| = \frac{\sqrt{2}}{3} \text{ unit}$$

Exercise 6.2 :

1) Find the length of the perpendicular from (2,–3,1) to the line $\dfrac{x+1}{2} = \dfrac{y-3}{3} = \dfrac{z+2}{-1}$.

2) Find the co-ordinates of the foot of the perpendicular drawn from the point $2\hat{i}-\hat{j}+5\hat{k}$ to the line $\overline{r} = (11\hat{i}-2\hat{j}-8\hat{k}) + \lambda(10\hat{i}-4\hat{j}-11\hat{k})$. Also find the length of the perpendicular.

3) Find the perpendicular distance of the point (1,0,0) from the line $\frac{x-1}{2}=\frac{y+3}{-3}=\frac{z+10}{8}$. Also find the co-ordinates of the foot of the perpendicular.

4) By computing the shortest distance, determine whether following lines intersect each other.

i) $\bar{r} = (\hat{i}-\hat{j}) + (2\hat{i}+\hat{k})$ and $\bar{r} = (2\hat{i}-\hat{j}) + \mu(2\hat{i}+\hat{j}-\hat{k})$.

ii) $\frac{x-5}{4}=\frac{y-7}{-5}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-7}{1}=\frac{z-5}{3}$

5) If lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect each other then find k.

Ex. 1) Find the vector equation of the plane passing through the point having position vector $2\hat{i}+3\hat{j}+4\hat{k}$ and perpendicular to the vector $2\hat{i}+\hat{j}+2\hat{k}$.

Solution :

We know that the vector equation of the plane passing through $A(\bar{a})$ and normal to vector $\bar{n}$ is given by $\bar{r}\cdot\bar{n} = \bar{a}\cdot\bar{n}$

Here $\bar{a}_1 = 2\hat{i}-3\hat{j}+4\hat{k}$, $\bar{n} = 2\hat{i}-\hat{j}+2\hat{k}$

$\bar{a}\cdot\bar{n} = (2\hat{i}-3\hat{j}+4\hat{k})\cdot(2\hat{i}-\hat{j}+2\hat{k}) = 4+3-8=-1$

The vector equation of the plane is $\bar{r}\cdot(2\hat{i}-\hat{j}+2\hat{k}) = -1$

Ex. 2) Find the Cartesian equation of the plane passing through $A(1, 2, 3)$ and the direction ratios of whose normal are 3, 2, 5.

Solution :

The equation of the plane passing through $A(x_1, y_1, z_1)$ and normal to the line having direction ratios a,b,c is $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

Here $(x_1, y_1, z_1) \equiv (1,2,3)$ and direction ratios of the normal are 3,2,5.

The Cartesian equation of the plane is $3(x-1)+2(y-2)+5(z-3)=0$

$3x+2y+5z-22=0$

Ex. 3) The foot of the perpendicular drawn from the origin to a plane is $M(2,1,-2)$. Find vector equation of the plane.

Solution :

OM is normal to the plane.

$\therefore$ The direction ratios of the normal are $2,1,-2$.

The plane passes through the point having position vector $2\hat{i}+\hat{j}-2\hat{k}$ and vector $\overline{OM}=$ $= 2\hat{i}+\hat{j}-2\hat{k}$ is normal to it.

Its vector equation is $\overline{r}\cdot\overline{n} = \overline{a}\cdot\overline{n}$

$\overline{r}\cdot(2\hat{i}+\hat{j}-2\hat{k})=(2\hat{i}+\hat{j}-2\hat{k})\cdot(2\hat{i}+\hat{j}-2\hat{k})$

$\overline{r}\cdot(2\hat{i}+\hat{j}-2\hat{k})=9$

Ex. 4) Find the vector equation of the plane passing through the point A(–1, 2, –5)an parallel to vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $\hat{i}+\hat{j}-\hat{k}$.

Solution :

The vector equation of the plane passing through point A($\overline{a}$) and parallel to $\overline{b}$ and $\overline{c}$ is

$\overline{r}.(\overline{b}\times\overline{a}) = \overline{a}.(\overline{b}\times\overline{c})$

Here $\overline{a}=-\hat{i}-2\hat{j}-5\hat{k}$, $\overline{b}=4\hat{i}-\hat{j}+3\hat{k}$, $\overline{c}=\hat{i}-\hat{j}-\hat{k}$.

$$\overline{b}\times\overline{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ 1 & 1 & -1 \end{vmatrix} = 2\hat{i}-7\hat{j}+5\hat{k}$$

$\overline{a}\ (\overline{b}\times\overline{c}) = (\hat{i}-2\hat{j}+5\hat{k})\cdot(-2\hat{i}+7\hat{j}+5\hat{k}) = -9$

The required equation is $\overline{r}\cdot(-2\hat{i}+7\hat{j}+5\hat{k}) = -9$

Ex. 5) Find the Cartesian equation of the plane $\overline{r}=(\hat{i}-\hat{j})+\lambda(\hat{i}+\hat{j}+\hat{k})+\mu(\hat{i}+2\hat{j}+3\hat{k})$

Solution :

Given plane is perpendicular to vector $\overline{n}$, where

$$\overline{n} = \overline{b}\times\overline{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\hat{i}+2\hat{j}+3\hat{k}$$

$\therefore$ The direction ratios of the normal are 5,–2,–3.

And plane passes through $A(1,-1,0)$.

$\therefore$ Its Cartesian equation is $a(x-x_1)+b(y-y_2)+c(z-z_3) = 0$

$\therefore\ 5(x-1)-2(y+1)-3(z-0)=0$

$\therefore\ 5x-2y-3z-7=0$

Ex. 6) Find the vector equation of the plane passing through points $A(1,1,2)$, $B(0,2,3)$ and $C(4,5,6)$.

Solution :

Let $\overline{a}$, $\overline{b}$ and $\overline{c}$ be position vectors of points A, B and C respectively.

$\therefore\ \overline{a} = \hat{i}+\hat{j}+2\hat{k}$, b=2j+3k, c=$4\hat{i}+5\hat{j}+6\hat{k}$.

$\therefore\ \overline{b}-\overline{a}=-\hat{i}+\hat{j}+\hat{k}$ and $\overline{c}-\overline{a}=3\hat{i}+4\hat{j}+4\hat{k}$

$$\therefore\ (\overline{b}-\overline{a})\times(\overline{c}-\overline{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 3 & 4 & 4 \end{vmatrix} 7\hat{j} -7\hat{k}$$

The plane passes through $\overline{a}= \hat{i}+\hat{j}+2\hat{k}$ and $7\hat{j} -7\hat{k}$ is normal to the plane.

$\therefore$ Its equation is $(\overline{r}-\overline{a})\cdot(7\hat{j}-7\hat{k})=0$

$\therefore\ [\overline{r}-(\hat{i}+\hat{j}+2\hat{k})]\cdot(7\hat{j}-7\hat{k})=0$

$\therefore\ \overline{r}.(7\hat{j}-7\hat{k})=(\hat{i}+\hat{j}+2\hat{k}).(7\hat{j}-7\hat{k})$

$\therefore\ \overline{r}.(7\hat{j}-7\hat{k})= -7$ is the required equation.

Ex. 7) Find the vector equation of the plane which is at a distance of 6 unit from the origin and to which the vector $2\hat{i}-\hat{j}+2\hat{k}$ is normal.

Solution :

Here $p=6$ and $\overline{n} = 2\hat{i}-\hat{j}+2\hat{k}$ $\quad \therefore\ |\overline{n}| =3$

$$\therefore\ \hat{n}=\frac{\overline{n}}{|\overline{n}|} = \frac{2\hat{i}-\hat{j}+2\hat{k}}{3}$$

The required equation is $\overline{r}\cdot\hat{n}=p$

$$\therefore\ \overline{r}\cdot\left(\frac{2\hat{i}-\hat{j}+2\hat{k}}{3}\right)$$

$$\therefore\ \overline{r}\cdot(\hat{i}-\hat{j}+2\hat{k}) = 18$$

Ex. 8) Find the perpendicular distance of the origin from the plane $x-3y+4z-6=0$.

Solution :

First we write the given Cartesian equation in normal form.

i.e. in the form $lx+my+nz=p$

Direction ratios of the normal are $1,-3,4$.

$\therefore$ direction cosines are $\dfrac{1}{\sqrt{26}}, \dfrac{-3}{\sqrt{26}}, \dfrac{4}{\sqrt{26}}$

Given equation can be written as $\frac{1}{\sqrt{26}}x - \frac{3}{\sqrt{26}}y + \frac{4}{\sqrt{26}}z = \frac{6}{\sqrt{26}}$

$\therefore$ The distance of the origin from the plane is $\frac{6}{\sqrt{26}}$.

Ex. 9) Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane $2x+y-2z=18$

Solution :

The Direction ratios of the normal to the plane $2x+y-2z=18$ are 2,1,–2.

$\therefore$ direction cosines are $\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$.

$$\therefore\ l = \frac{2}{3},\ m = \frac{1}{3},\ n = -\frac{2}{3}$$

The normal form of the given Cartesian equation is $\frac{2}{3}x + \frac{1}{3}y - \frac{2}{3}z = 6$

$$\therefore\ p = 6$$

The coordinates of the foot of the perpendicular

$(lp,mp,np) \equiv \left[6\left(\frac{2}{3}\right), 6\left(\frac{1}{3}\right), 6\left(-\frac{2}{3}\right),\right] \equiv (4, 2, -4)$

Ex. 10) Reduce the equation $\overline{r}\cdot(3\hat{i} - 4\hat{j} + 12\hat{k}) = 8$ to the normal form and hence find

i) the length of the perpendicular from the origin to the plane

ii) direction cosines of the normal.

Solution :

Here $\overline{n} = 3\hat{i} - 4\hat{j} + 12\hat{k}\ \therefore\ |\overline{n}| = 13$

The required normal form is $\overline{r}\cdot\frac{(3\hat{i} - 4\hat{j} + 12\hat{k})}{13} = \frac{8}{13}$

i) the length of the perpendicular from the origin to the plane is $\frac{8}{13}$

ii) direction cosines of the normal are $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$.

Exercise 6.3 :

1) Find the vector equation of a plane which is at 42 unit distance from the origin and which is normal to the vector $2\hat{i}+\hat{j}-2\hat{k}$.
2) Find the perpendicular distance of the origin from the plane $6x-2y+3z-7=0$.
3) Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x+6y-3z=63$.
4) Find the vector equation of the plane passing through the point having position vector $\hat{i}+\hat{j}-\hat{k}$ and perpendicular to the vector $4\hat{i}+5\hat{j}+6\hat{k}$.
5) Find the Cartesian equation of the plane passing through $A(-1,2,3)$ and the direction ratios of whose normal are 0,2,5.
6) Find the Cartesian equation of the plane passing through $A(7,8,6)$ and parallel to the XY plane.

Ex. 1) Find the angle between planes $\overline{r}\cdot(\hat{i}-\hat{j}+2\hat{k}) = 8$ and $\overline{r}\cdot(-2\hat{i}-\hat{j}+\hat{k}) = 3$.

Solution :

Normal to the given planes are $\overline{n}_1 = \hat{i}+\hat{j}-2\hat{k}$ and $\overline{n}_2 = -2\hat{i}+\hat{j}+\hat{k}$

The acute angle θ between normals is given by

$$\cos\theta = \left|\frac{\overline{n}_1 \cdot \overline{n}_1}{|\overline{n}|_1\ |\overline{n}|_1}\right|$$

$$\therefore\ \cos\theta = \left|\frac{(\hat{i}-\hat{j}+2\hat{k})\cdot(-2\hat{i}+\hat{j}+\hat{k})}{\sqrt{6}\ \sqrt{6}}\right| = \frac{-3}{6} = \frac{1}{2}$$

$$\therefore\ \cos\theta = \frac{1}{2}\ \ \therefore\theta = 60^\circ = \frac{\pi}{3}$$

The acute angle between normals $\overline{n}_1$ and $\overline{n}_2$ is 60°

$\therefore$ The angle between given planes is 60°

Ex. 2) Find the angle between the line $\overline{r}\cdot(\hat{i}-\hat{j}+\hat{k}) + \lambda(\hat{i}-\hat{j}+\hat{k})$ and the plane $\overline{r}\ .\ (2\hat{i}-\hat{j}+\hat{k}) = 8$

Solution :

The angle between the line $\overline{r} = \overline{a},\lambda\overline{b}$ and the plane $\overline{r}\,.\,\overline{n} = d$ is given by $\sin\theta = \left|\frac{\overline{b}\ .\ \overline{n}}{|\overline{b}|\ |\overline{n}|}\right|$

Here $\overline{b} = \hat{i}+\hat{j}+\hat{k}$ and $\overline{n} = 2\hat{i}-\hat{j}+\hat{k}$

$\therefore\ \overline{b}\,.\,\overline{n} = (\hat{i}+\hat{j}+\hat{k})\cdot(2\hat{i}+\hat{j}+\hat{k}) = 2-1+1=2$

$|\overline{b}| = \sqrt{1+1+1} = \sqrt{3}$ and $|\overline{n}| = \sqrt{4+1+1} = \sqrt{6}$

$$\therefore\ \sin\theta = \left|\frac{\overline{b}\ .\ \overline{n}}{|\overline{b}|\ |\overline{n}|}\right| = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3}$$

$$\therefore\ \sin\theta^{-1} = \left(\frac{\sqrt{2}}{3}\right)$$

Ex. 3) Find the distance of the point $4\hat{i}-3\hat{j}+2\hat{k}$ from the plane $\overline{r}\cdot(-2\hat{i}-\hat{j}+2\hat{k}) = 6$.

Solution :

Here $\overline{a} = 4\hat{i}+3\hat{j}+2\hat{k}$, $\overline{n} = 2\hat{i}+\hat{j}+2\hat{k}$

$\therefore\ |\overline{n}| = \sqrt{(2-0)^2+(1)^2+(-2)^2 = 3}$

$$\therefore\ \hat{n} = \frac{\sqrt{2\hat{i}+\hat{j}+2\hat{k}}}{3}$$

The normal form of the equation of the given plane is

$$\overline{r}. = \frac{\sqrt{2\hat{i}+\hat{j}+2\hat{k}}}{3} = 2 \qquad \therefore\ p = 2$$

$$\text{Now, } \overline{a}.\ \hat{n} = (4\hat{i}-3\hat{j}+2\hat{k}).\ \frac{(2\hat{i}+\hat{j}+2\hat{k})}{3}$$

$$\frac{(4\hat{i}-3\hat{j}+2\hat{k})\cdot(-2\hat{i}+\hat{j}-2\hat{k})}{3} \quad -\frac{15}{3} = -5$$

$\therefore\ |\overline{a}\,.\,\hat{n} = -5$

The required distance is given by $|p-\overline{a}.\hat{n}\ |=|2-(-5)|=7$

Therefore the distance of the point $4\hat{i}-3\hat{j}+2\hat{k}$ from the plane $\overline{r}.\ (-2\hat{i}+\hat{j}-2\hat{k})=6$ is 7 unit.

Exercise 6.4

1) Find the angle between planes $\overline{r}.(\hat{i}+\hat{j}+2\hat{k})=13$ and $\overline{r}.(2\hat{i}-\hat{j}+\hat{k})=31$.

2) Find the acute angle between the line $\overline{r}.=(\hat{i}+2\hat{j}+2\hat{k})+(2\hat{i}+3\hat{j}-6\hat{k})$ and the planer $\overline{r}.(2\hat{i}-\hat{j}+\hat{k})=0$.

3) Find the distance of the point $4\hat{i}-3\hat{j}+\hat{k}$ from the plane $\overline{r}.(2\hat{i}+3\hat{j}-6\hat{k})=21$.

4) Find the distance of the point (1,1,–1)from the plane $3x+4y-12z+20=0$.

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7. Linear Programming Problem

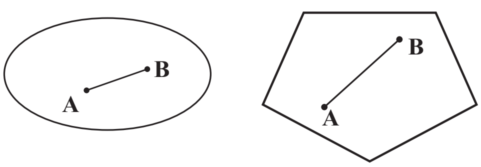
Linear inequation in two variables :

A linear inequation in two variables x, y is a mathematical expression of the form $ax + by < c$ or $ax + by > c$ where $a \neq 0$, $b \neq 0$ simultaneously and $a, b \in R$.

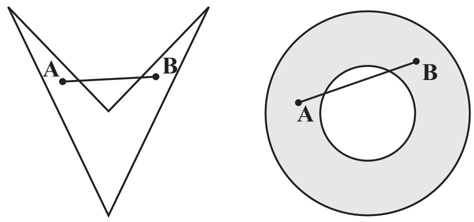
Convex set :

A set of points in a plane is said to be a convex set if line segment joining any two points of the set entirely lies within the set.

he following sets are convex sets :



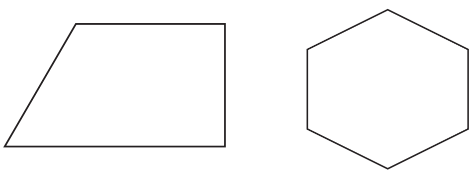
The following sets are not convex sets :



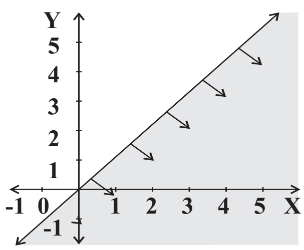
Note:

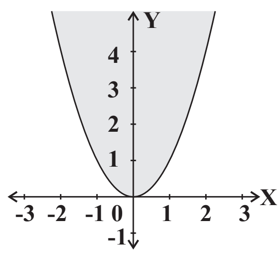
i) The convex sets may be bounded.

Following are bounded convex sets.



ii) Convex sets may be unbounded.



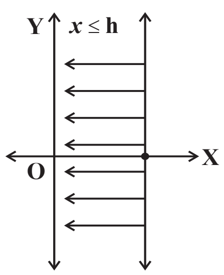


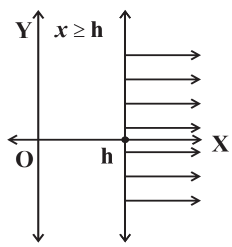
Note :

1) Graphical representations of $x \leq h$ and $x \geq h$ On the Cartesian coordinate system.

Draw the line x = h in XOY plane.

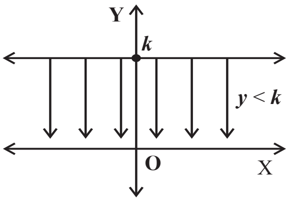
The solution set is the set of points lying on the Left side or Right side of the line $x = h$.

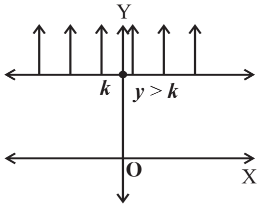




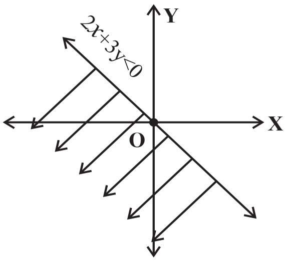
2) Graphical representation of y ≤ k and y ≥ k on the Cartesian coordinate system. Draw the line y = k in XOY plane.

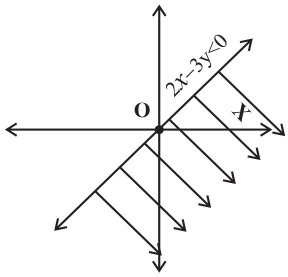
The solution set is the set of points lying below or above the line y = k





3) Graphical representation of ax + by ≤ 0 and ax + by ≥ 0 on the Cartesian coordinate system. The line ax c by = 0 passes through the origin, see the following graphs.



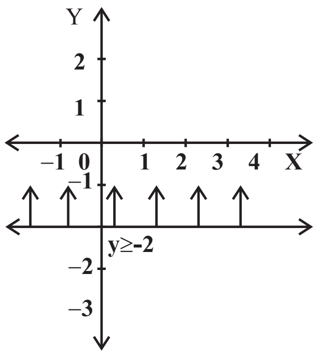


Ex. 1. Show the solution sets for the following in equations graphically.

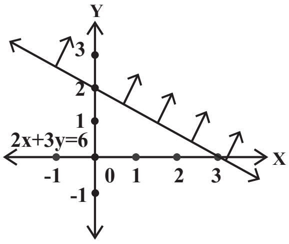
a) y ≥ − 2 b) 2 x 3y ≥ 6

Solution :

a) To draw : y ≥ − 2 ; Draw line y = − 2



b) To draw : $2x + 3y \geq 6$; Draw line : $2x + 3y = 6$



x	Y	(x, y)
3	0	(3, 0)
0	2	(0, 2)

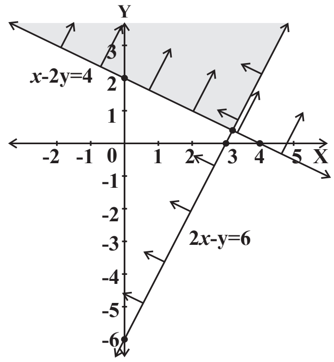
$2x + 3y$ (0, 0) = 0 < 6. Therefore, the required region is the non-origin side of the line.

Ex. 2 : Find the common region of the solutions of the inequations $x + 2y \geq 4, 2x - y \leq 6$.

Solution : To find the common region of : $x + 2y \geq 4$ and $2x - y \leq 6$

Draw the lines : $x + 2y = 4$ and $2x - y = 6$

Equation of Line	X	Y	Line passes through (x, y)	Sign	Region
$x + 2y = 4$	4	0	(4, 0)	$\geq$	Non-origin side
	0	2	(0, 2)		
$2x - y = 6$	3	0	(3, 0)	$\leq$	Origin side
	0	–6	(0, –6)		



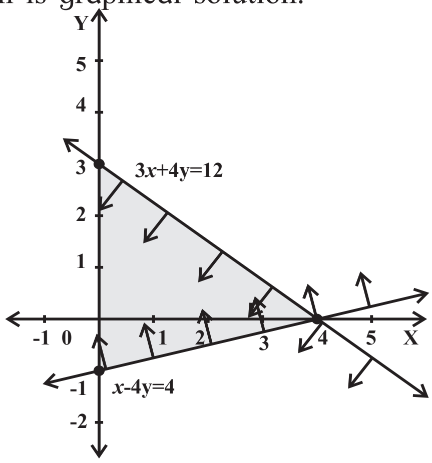
Ex. 2. Find the graphical solution of $3x + 4y \leq 12$, and $x - 4y \leq 4$

Solution : To find the graphical solutions of : $3x + 4y \leq 12$ and $x - 4y \leq 4$

Draw the lines $L_1 : 3x + 4y = 12$ and $L_2 : x - 4y = 4$

Equation of Line	X	Y	Line passes through (x, y)	Sign	Region
$3x + 4y = 12$	4	0	(4, 0)	$\geq$	Origin side
	0	3	(0, 3)		
$X - 4y = 4$	4	0	(4, 0)	$\leq$	Origin Side
	0	–1	(0, -1)		

The common shaded region is graphical solution.



Exercise 7.1

1) Solve graphically :

i) $x \geq 0$ ii) $x \leq 0$ iii) $2x - 3 \geq 0$ iv) $5y + 3 \leq 0$

v) $x + 2y \leq 6$ vi) $5x - 3y \leq 0$

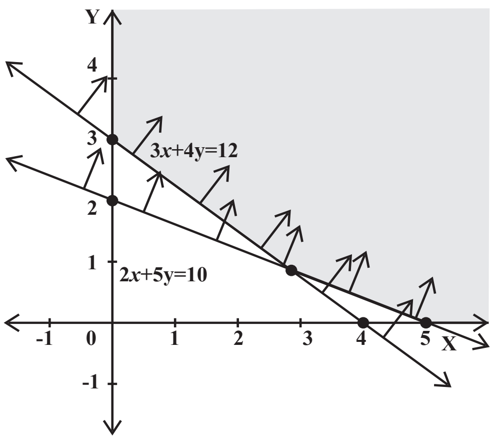
2) Solve graphically : i) $2x + y \geq 2$ and $x - y \leq 1$ ii) $x - y \leq 2$ and $x + 2y \leq 8$

Definition : A solution which satisfies all the constraints is called a feasible solution.

Ex 1. Find the feasible solution of the system of in equations $3x + 4y \geq 12$, $2x + 5y \geq 10$, $x \geq 0$, $y \geq 0$

Equation of Line	Draw line	X	Y	Line passes through (x, y)	Sign	Region
$3x + 4y = 12$	$L_1 : 3x + 4y = 12$	4	0	(4, 0)	$\geq$	Origin side
		0	3	(0, 3)		
$X - 4y = 4$	$L_2 : 2x + 5y = 10$	4	0	(4, 0)	$\leq$	Origin Side
		0	-1	(0, -1)		

Solutions : Common shaded region is the feasible solution.



Ex. 2 : A manufacturer produces two items A and B. Both are processed on two machines I and II. A needs 2 hours on machine 1 and 2 hours on machine. II B needs 3 hours on machine I and I hour on machine II. If machine I can run maximum 12 hours per day and II for 8 hours per day, construct a problem in the form of in equations and fined its feasible solution graphically.

Solution : Let x units of product A and y units of product B ne produced.

$x \geq 0$, $y \geq 0$

Tabular form is:

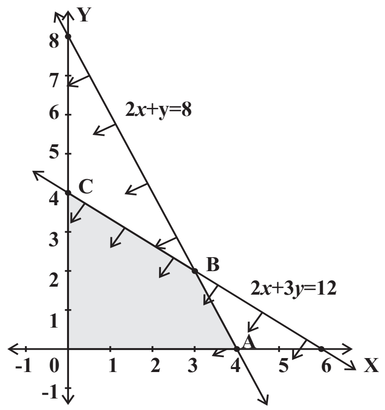
Machine	Product A (x)	Product B(y)	Availability
I	2	3	12
II	2	1	08

Inequations are $2x + 3y \leq 12$, $2x + y \leq 8$, $x \geq 0$, $y \geq 0$.

To draw graphs of the above inequations:

To draw	**Draw line**	**X**	**Y**	**Line passes through**	**Sign**	**Region lie on**
$2x + 3y \leq 12$	L_1 : $2x + 3y = 12$	0	4	(0, 4)	$\leq$	Origin side of Line L_1
		6	0	(6, 0)		
$2x + 4 \leq 8$	L_2	0	8	(0, 8)	$\leq$	Origin side of Line L_2
	$2x + 4 = 8$	4	0	(4, 0)		

The common shaded region OABCO the feasible region.



Exercise 7. 2

I) Find the feasible solution of the following inequations graphically.

1) Solve graphically : $x - y \leq 2$ and $x + 2y \leq 8$

2) Solve graphically : $x + y \geq 6$ and $x + 2y \leq 10$

3) Solve graphically : $2x + 3y \leq 6$ and $x + 4y \geq 4$

4) Solve graphically : $2x + y \geq 5$ and $x - y \leq 1$

5) Solve graphically : $2x + 3y \leq 6$, $x + y \geq 2$, $x \geq 0$, $y \geq 0$

6) Solve graphically : $2x + y \geq 2$ and $x - y \leq 1$

Linear Programming Problems (L.P.P.)

Linear Programming is used to minimize the cost of production and maximizing the profit. These problems are related to efficient use of limited resources like raw materials, man-power, availability of machine time and cost of the material and so on.

Linear Programming problems are also known as optimization problems. The mathematical programming involves optimization of a certain function, called objective function, subject to given conditions or restrictions known as constraints.

Meaning of L.P.P.

Linear implies all the mathematical functions contain variables of index of most one. A L.P.P. may be defined as the problem of maximizing or minimizing a linear function subject to linear constraints. These constraints may be equations or inequations.

Terms related to L.P.P.

1) **Decision variables :** The variables involved in L.P.P. are called decision variables.
2) **Objective function :** A linear function of variables which is to be optimized, i.e. either maximized or minimized, is called an objective function.
3) **Constraints :** Conditions under which the objective function is to be optimized are called constant. These constraints are in the form of equations or inequations.
4) **Non-negativity constraints :** In some situations, the values of the variables under considerations may be positive or zero due to the imposed constraints. Such constraints are referred as non-negativity constraints.

Mathematical formulations of L.P.P.

Step 1) : Identify the decision variables (x, y) or (x_1, x_2)

Step 2) : Identify the objective function and write it as mathematical expression in terms of decision variables.

Step 3) : Identify the different constraints and express them as mathematical equations/ inequations.

Note : i) We shall study L.P.P. with at most two variables

ii) We shall restrict ourselves to L.P.P involving non-negativity constraints.

Ex. 1 : A Toy manufacture produces bicycles and tricycles, each of which must be processed through two machine A and B, Machine A has maximum of 120 hours available and machine B has a maximum of 180 hours available. Manufacturing a tricycle required 6 hours on machine A and 3 hours on machine B. If profits are Rs. 65 got s bicycle and Rs. 45 for a tricycle, formulate L. P.P to have maximum profit.

Solution : Let z be the profit, which can be made by manufacturing and selling x tricycles and y bicycles. $x \geq 0$, $y \geq 0$

Total Profit $z = 45x + 65y$ $\therefore$ Maximize $z = 45x + 65y$

It is given that

Machine	Tricycles (x)	Bicycles (y)	Availability
A	6	4	120
B	3	10	180

From the above, remaining conditions are $6x + 4y \leq 120$, $3x + 10y \leq 180$

The required formulated L.P.P. is as follows : Maximize $z = 45x + 65y$, Subject to the constraints

$6x + 4y \leq 120$, $3x + 10y \leq 180$, $x \geq 0$, $y \geq 0$

Ex. : 2 A company manufactures two types of toys A and B. Each toy of type A requires 2 minutes for cutting and 1 minute for assembling. Each toy of type B requires 3 minutes for cutting and 4 minute for assembling. There are 3 hours available for cutting and 2 hours are available for assembling. On selling a toy of type A the company gets a profit of Rs.10 and that on toy of type B is Rs.20. Formulate the L.P.P. to maximize profit.

Solution : Suppose, the company manufactures x toys of type A and y toys of type B. Means, $x \geq 0$, $y \geq 0$

Let P be the total profit

On selling a toy of type A, company gets Rs. 10 and that on a toy of type B is Rs. 20

$\therefore$ total profit on selling x toys of type A and y toys of type B is $p = 10x + 20y$.

$\therefore$ maximize $p = 10x + 20y$:

The conditions are : $2x + 3y \leq 80$, $x + 4y \leq 10$, $x \geq 0$, $y \geq 0$

Formal definitions related to L.P.P.

1) **Solution of L.P.P. :** A set of values of the decision variables $x_1, x_2, \dots\dots\dots xn$ which satisfy the conditions of given linear programming problem is called a solution to that problem.
2) **Feasible solution :** A solution which satisfies the given constraints is called a feasible solution
3) **Optimal feasible solution :** A feasible solution which maximizes or minimizes the objective function as per the requirements is called an optimal feasible solution.
4) **Feasible region :** The common region determined by all the constraints of the L.P.P. is called the feasible region.

Solution of L.P.P : Graphical method

A set of values of the variables which satisfies all the constraints of the L.P.P. is called the solution of the L.P.P.

Optimum feasible solution :

A feasible solution which optimizes (either maximizes or minimizes) the objective function of L.P.P. is called optimum feasible solution

Theorem 1 : The set of all feasible solutions of L.P.P. is a convex set.

Theorem 2 : The objective function of L.P.P. attains its optimum value (either maximum or minimum) at least at one of the vertices of convex polygon. This is known as convex polygon theorem.

Corner – Point Method :

1) Convent all inequations of the constraints into equations.
2) Draw the lines in X – Y plane
3) Locate common region indicated by the constraints. This common region is feasible region.
4) Find the vertices of feasible region
5) Find the value of the objective function z at all vertices of feasible region.

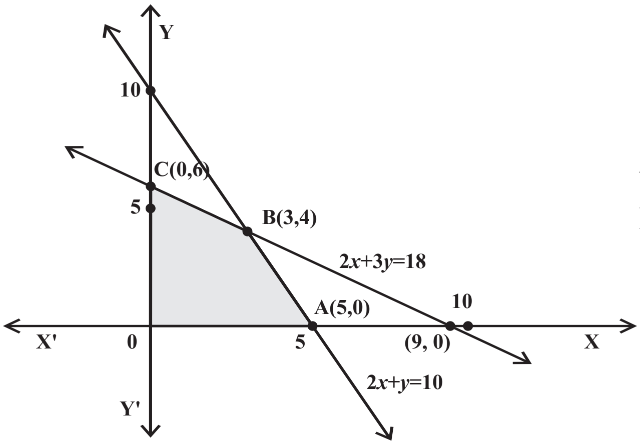
Solve graphically the following Linear Programming Problems:

Example 1 : Maximize : $z = 9x + 13y$ subject to $2x + 3y \leq 18$, $2x + y \leq 10$, $x \geq 0$, $y \geq 0$.

Solution : To draw $2x + 3y \leq 18$ and $2x + y \leq 10$

Draw line $2x + 3y = 18$ and $2x + y = 10$

To draw	x	y	Line passes through	Sign	Region lie on
L_1 $2x + 3y = 18$	0	6	(0, 6)	$\leq$	Origin side of Line L_1
	9	0	(9, 0)		
L_2 $2x + y = 10$	0	10	(0, 10)	$\leq$	Origin side of Line L_2
	5	0	(5, 0)		



The common shaded region is feasible region with vertices OA (0, 6) B (1, 5), C (0, 10), Form the table, minimum value of z = 15, occurs at B (1, 5) i.e. when $x = 1$, y = 5

(x, y) Vertex of S	Value of z = 9x + 13y at (x, y)
O(0, 0)	0
A (5, 0)	45
B (3, 4)	79
C (0, 6)	78

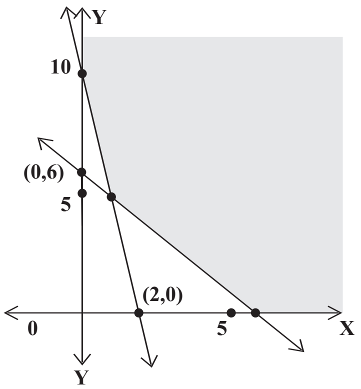
From the table, maximum value of z = 79, occurs at B (3, 4) i.e., when x = 3, y = 4.

Example 2 : Minimize : $z = 5x + 2y$ subject to $5x + y \geq 10$, $x + y \geq 6$, $x \geq 0$, $y \geq 0$.

Solution : To draw $5x + y \geq 10$ and $x + y \geq 6$

Draw line $5x + y = 10$ and $x + y = 6$

To draw	x	y	Line passes through	Sign	Region lie on
L_1 : $5x + y = 10$	0	10	(0, 6)	≤	Origin side of Line L_1
	2	0	(9, 0)		
L_2 $2x + y = 10$	0	6	(0, 10)	≤	Origin side of Line L_2
	6	0	(5, 0)		



The common shaded region is feasible region with vertices A (6, 0), B (1, 5), C (0, 10),

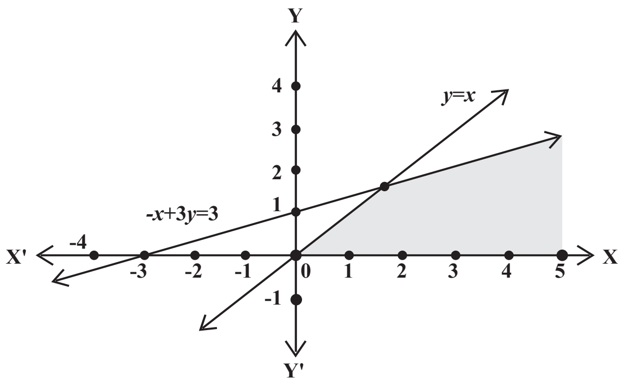
Form the table, minimum value of z = 15, occurs at B (1, 5) i.e. when $x = 1$, y = 5

(x, y) Vertex of S	Value of $z = 5x + 2y$ at (x, y)
A (6, 0)	30
B (1, 5)	15
C (0, 10)	20

Example 3 : Maximize $z = 3x + 4y$ subject to $x - y \geq 0$, $-x + 3y \leq 3$, $x \geq 0$, $y \geq o$

Solution To draw $x - y \geq 0$ and $-x + 3y \leq 3$. Draw line $x - y = 0$ and $-x + 3y = 3$

To draw	Draw line	X	Y	Line passes through	Sign	Region lie on
$x - y \geq 0$	$L_1 : x = y$	0	0	(0, 0)	$\geq$	A side
		1	1	(1, 1)		
$-x + 3y \leq 3$	$L_2 : -x + 3y = 3$	0	1	(0, 1)	$\leq$	Origin side of Line L_2
		–3	0	(–3, 0)		



From graph, we can see that the common shaded area is the feasible region which is unbounded (not a polygon). In such cases, the iso-profit lines can be moved away form the origin indefinitely. There is no finite maximum value of z within the feasible region.

Example 4 : Maximize : $z = 5x + 2y$ subject to $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x \geq 0$, $y \geq 0$.

Solution : To draw $3x + 5y \leq 15$ and $5x + 2y \leq 10$, Draw line $3x + 5y = 15$ and $5x + 2y = 10$

To draw	X	Y	Line passes through (x, y)	Sign	Region lies on
$L_1 : 3x + 5y = 15$	5	0	(5, 0)	$\leq$	Origin side of Line L_1
	0	3	(0, 3)		
$L_2 : 5x + 2y = 10$	2	0	(2, 0)	$\leq$	Origin side of Line L_2
	0	5	(0, 5)		

The shaded region O A B C is the feasible region with the vertices

O (0, 0), A (2, 0), B $\left(\frac{20}{19}, \frac{45}{19}\right)$, C (0, 3)

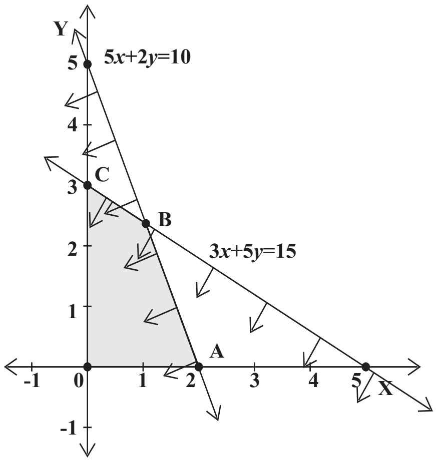
$Z_0 = 0$, $Z_A = 10$, $Z_B = 10$, $Z_c = 6$

Maximum value of z occurs at A and B and is z = 10

Maximum value of z occurs at every point lying on the segment AB

Hence there are infinite number of optimal solutions

Note : If the two distinct points produce the same minimum value then the minimum value of objective function occurs at every point on the segment joining them.



Exercise 7.3

Solve the following L.P.P. by graphical method :

1) Maximize : $z = 11x + 8y$ subject to $x \leq 4$, ≤ 6, $x + y \leq 6$, $x \geq 0$, $y \geq 0$

2) Maximize : $z = 4x + 6y$ subject to $3x + 2y \leq 12$, $x + y \geq 4$, $x, y \geq 0$

3) Maximize = $z = 7x + 11y$ subject to $3x + 5y \leq 26$, $5x + 3y \leq 30$, $x \geq 0$, $y \geq 0$

4) Maximize : $z = 10x + 25y$ subject to $0 \leq x \leq 3$, $0 \leq y \leq 3$, $x + y \leq 5$ also find maximum value of z.

5) Maximize : $z = 3x + 5y$ subject to $x + 4y \leq 24$, $3x + y \leq 21$, $x + y \leq 9$, $x \geq 0$, $y \geq 0$

6) Minimize : $z = 7x + y$ subject to $5x + y \geq 5$, $x + y \geq 3$, $x \geq 0$, $y \geq 0$.

7) Minimize : $z = 8x + 10y$ subject to $2x + y \geq 7$, $2x + 3y \geq 15$, $y \geq 2$, $x \geq 0$, $y \geq 0$

8) Minimize : $z = 6x + 21y$ subject to $x + 2y \geq 3$, $4y \geq 4$, $3x + y \geq 3$, $x \geq 0$, $y \geq 0$

Answer Key (Part - I)

Exercise 1.1

1) i) p : A triangle is equilateral, q : A triangle is equiangular. Symbolic form : $p \leftrightarrow q$

   ii) p : Price increases, q : Demand falls. Symbolic form : $p \wedge q$

2) i) T ii) T iii) T

3) i) $(p \wedge q) \vee F$ ii) Madhuri has curly hair or brown eyes. iii) $\sim p \vee (q \wedge t)$

   iv) 'Shweta is a doctor and Seema is a teacher.' v) $p \vee \sim p \equiv T$

Exercise 1.2

1) i) T ii) T iii) T iv) T v) T vi) T

2) i) $\exists\ n \in N$ such that $n + 7 \leq 6$

   ii) The kitchen is not neat or it is not tidy

   iii) Some students of this college do not live in the hostel

   iv) 6 is not an even number and 36 is not a perfect square

   v) Diagonals of a parallelogram are perpendicular but it is not a rhombus.

   vi) Mangos are not delicious or they are not expensive.

   vii) A person is rich and not a software engineer or a person is a software engineer and not rich.

Exercise 1.3

1) i) $\sim (p \vee q) \equiv \sim p \wedge \sim q$ (FFFT)

   ii) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ (TFFT)

   iii) $p \wedge q \equiv \sim (p \rightarrow \sim q)$ (TFFF)

   iv) $\sim p \wedge q \equiv (p \vee q) \wedge \sim p$ (FFTF)

   v) $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$ (TFTTTTTT)

   vi) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$ (TFFT)

   vii) $(p \wedge q) \vee \sim q \equiv p \vee \sim q$ (TTFT)

2) i) $(p \to q) \leftrightarrow (\sim p \vee q)$ TTTT, Tautology

ii) $(p \wedge \sim q) \leftrightarrow (p \to q)$ FFFF, Contradiction

iii) $\sim (\sim p \sim q)\ q$ TTTF, Contingency

iv) $(p \wedge q) \vee (p \wedge r)$ TTTFFFFF, Contingency

v) $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$ TTTTTTTT, Tautology

vi) $[(p \to q) \wedge q] \to p$ TTFT, Contingency

3) i) 'If two triangles are congruent then their areas are equal.'

Converse is $q \to p$: If areas of two triangles are equal then they are congruent.

Inverse is $\sim p \to \sim q$: If two triangles are not congruent then their areas are not equal.

Contra positive is $\sim q \to \sim p$: If areas of two triangles are not equal then they are not congruent.

ii) If it rains then the match will be cancelled."

Converse is $q \to p$: If the match is cancelled then it rains.

Inverse is $\sim p \to \sim q$: If it does not rain then the match will not be cancelled.

Contra positive is $\sim q \to \sim p$: If the match is not cancelled then it does not rain.

iii) 'If an angle is right angle then its measure is 90^0'

Converse is $q \to p$: If the measure of an angle is 90^0, then it is a right angle.

Inverse is $\sim p \to \sim q$: If an angle is not a right angle, then its measure is not 90^0

Contra positive is $\sim q \to \sim p$: If the measure of an angle is not 90^0, then it is not a right angle.

iv) 'If a sequence is bounded, then it is convergent.'

Converse is $q \to p$: If a sequence is convergent, then it is bounded.

Inverse is $\sim p \to \sim q$: If a sequence is not bounded, then it is not convergent

Contra positive is $\sim q \to \sim p$: If a sequence is not convergent, then it is not bounded.

v) 'If $x < y$ then $x^2 < y^2$'

Converse is $q \to p$: If $x^2 < y^2$ then $x < y$

Inverse is $\sim p \to \sim q$: If $x \geq y$ then $x^2 \geq y^2$

Contra positive is $\sim q \to \sim p$: If $x^2 \geq y^2$ then $x \geq y$

Exercise 2.1

1) i) $A \sim \begin{bmatrix} 4 & 5 \\ 3 & 1 \end{bmatrix}$ ii) $B \sim \begin{bmatrix} -1 & -6 & -1 \\ 2 & 5 & 4 \end{bmatrix}$

iii) $A \sim \begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 3 \end{bmatrix}$ iv) $A \sim \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 9 & 9 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 15 & 12 & 3 \end{bmatrix}$

2) i) $k = -3$

3) $\lambda = \pm 1$

4) ii) and iii) invertible.

5) $AB = \begin{bmatrix} 11 & 3 \\ 3 & 2 \end{bmatrix}$ and $(AB)^{-1} = \begin{bmatrix} 1 & -3 \\ -7 & 11 \end{bmatrix}$

6) -

7) i) $A_{11} = 4,\ A_{12} = 3,\ A_{21} = -2,\ A_{22} = -1$

ii) $A_{11} = -1,\ A_{12} = -4,\ A_{21} = -3,\ A_{22} = 1$

8) i) $\begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$ ii) $\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ 2 & 0 & 1 \end{bmatrix}$

9) $\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

10) i) $\begin{bmatrix} \frac{-2}{17} & \frac{5}{17} \\ \frac{3}{17} & \frac{1}{17} \end{bmatrix}$ ii) $\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$

11) $\begin{bmatrix} 13 & 2 & -7 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

12) i) $x = 4,\ y = -3$ ii) $x = 1,\ y = 1,\ z = 1$

13) i) $x = 0,\ y = 1$ ii) $x = -3,\ y = 4,\ z = -2$

Exercise 3.1

1) 1) $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{4}$ 2) $\theta = \frac{\pi}{6}$ and $\theta = \frac{7\pi}{6}$,

3) $\theta = \frac{5\pi}{6}$ and $\theta = \frac{11\pi}{6}$ 4) $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$

2) 1) $\theta = n\pi + (-1)^n \frac{\pi}{3}$ 2) $\theta = 2n\pi \pm \frac{\pi}{6}$

3) $\theta = n\frac{\pi}{3}$ or $\theta = n$ 4) $\theta = n\pi \pm \frac{\pi}{6},\ n \in Z$

5) $\theta = n\pi$ or $\theta = n\pi \pm 0 = 2n\pi$ 6) $\theta = \frac{n\pi}{2}$ or $\theta = \frac{n\pi}{2} + \frac{3\pi}{8}$

Exercise 3.2

1) $a : b : c = 1 : \sqrt{3} : 2$

Exercise 3.3

1) 1) $\frac{\pi}{4}$ 2) $\frac{2\pi}{3}$ 3) $\frac{3\pi}{5}$ 6) $x = \frac{1}{6}$ 7) $x = \frac{1}{\sqrt{3}}$

Exercise 4.1

1) $6x^2 + xy - y^2 = 0$ 2) $xy = 0$ 3) $x^2 - y^2 = 0$

4) $(x - 2y)(x + 2y) = 0$ 5) $k = 0$ 6) $k = 4$

7) Hint; show that $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

8) i) $6x^2 - 5xy + y^2 = 0$ ii) $3x^2 - y^2 = 0$ iii) $xy - 2x - y + 2 = 0$

9) $3x + 2y = 0$ and $x - 4y = 0$

10) $k = -1$ 11) 300

12) $3x^2 + 8xy + 6y^2 - 16 -13y + 5 = 0$

13) $x - y -3 = 0$ and $x - 2y - 4 = 0$

14) $K = -12$

15) $3x^2 + 8xy + 5y^2 = 0$

16) $k = \pm 2$

17) $25a + 16b = 40h$

19) $x^2 - 3y^2 = 0$

Exercise 5.1

1) $\bar{b} = -2\bar{a}$

2) Find $\overline{AB}$ $\overline{AC}$ and find relation.

3) $\frac{7}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$

4) (i) 6 (ii) 5

5) $\lambda = 3$

6) (i) 5 : 1 (ii) $p = -4$, $q = 9$

11) 60^0

12) $\frac{1}{3}$

13) $\frac{1}{9}(2\hat{i} - 2\hat{j} + \hat{k})$

14) $\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$

15) $0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

16) $\sqrt{168}$

20) 9 sq. units

21) 9

22) Use midpoint formula

23) 1

24) use $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

25) (0, 2, 3)

7) $p = -1, q = -3, r = \frac{7}{3}$

8) $-19\hat{i} + 8\hat{j} - 21\hat{k}$

9) 18

17) ± 60

18) $\sqrt{\frac{133}{2}}$ sq.units

19) 4, –4, 4

26) 110

27) $\frac{16}{3}$ cu. unit

Exercise 6.1

1) $\bar{r} = (-2\hat{i} + \hat{j} + \hat{k}) + \lambda\,(4\hat{i} - \hat{j} + 2\hat{k})$

2) $\bar{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda\,(3\hat{i} - 5\hat{j} + 8\hat{k})$

3) $\bar{r} = (5\hat{i} + 4\hat{j} + 3\hat{k}) + \lambda\,(-3\hat{i} + 4\hat{j} + 2\hat{k})$

4) $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda\,(2\hat{i} + \hat{j} - 3\hat{k})$

5) $\bar{r} = (-\hat{i} - \hat{j} + 2\hat{k}) + \lambda\,(3\hat{i} + 2\hat{j} + \hat{k})$

6) $\frac{x-2}{1} = \frac{y-2}{1} = \frac{z-1}{1}$

Exercise 6.2

1) $\frac{\sqrt{7434}}{14}$

2) $(1, 2, 3), \sqrt{14}$

3) $2\sqrt{6}$, 3, –4, –2

4) a) do not intersect b) do not intersect

5) $\frac{9}{2}$

Exercise 6.3

1) $\bar{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 126$

2) 1

3) $\left(\frac{18}{7}, \frac{54}{7}, \frac{-27}{7}\right)$

4) $\bar{r} \cdot (4\hat{i} + 5\hat{j} + 6\hat{k}) = 15$

5) $2y + 5z = 19$

6) $z = 6$

Exercise 6.4

1) 60^0

2) $\sin^{-1}\left(\frac{5}{7\sqrt{2}}\right)$

3) 4

4) $\frac{19}{13}$

Std. XII - Subject : Mathematics and Statistics
Part - I

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